Practice exercises

Vector Spaces. Change of basis – Survey

1. Analyze whether the following subsets are vector subspaces of the given ones. In affirmative case, obtain a basis for such subset.

   (a) $\mathcal{W} = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + 2z = 3x + y = 0\}$ of $\mathbb{R}^3$

   (b) $\mathcal{W} = \{M \in \mathcal{M}_{3\times 3}(\mathbb{R}) \mid M = M^T\}$ of $M_{3\times 3}(\mathbb{R})$

   (c) $\mathcal{W} = \{p(t) \in \mathbb{P}_2(\mathbb{R}) \mid 2p(0) - p(1) = 2\}$ of $P_2(\mathbb{R})$

2. Analyze if

   (a) $\mathcal{B} = \{(1, 2, -1), (0, 1, 1), (1, 0, -3)\}$ is a basis of $\mathbb{R}^3$

   (b) $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$

   (c) $\mathcal{B} = \{-x^2 + x + 5, 3x^2 + 5x + 3, 2x^2 + 2x - 1\}$ is a basis of $P_2(\mathbb{R})$

3. In the space of polynomials of degree, at most , 3 with real coefficients we consider the vectorial subspace spanned by the polynomials fulfilling the conditions

$$p'(1) = 0, \quad p'(2) = 0.$$

   Obtain a basis of this vectorial space.

4. Obtain the null space and the vectorial space generated by the columns of the following matrices:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 2 & -1 \\ 0 & 4 & 2 & 2 \\ 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

5. Given the vectors $v_1 = (1, 2, -1, 1)$, $v_2 = (2, 3, -1, 2)$, $v_3 = (1, 3, -2, 1)$ and $v_4 = (2, 1, 1, 2)$ of $\mathbb{R}^4$, we want:

   (a) To know if they form a basis of $\mathbb{R}^4$.

   (b) If not, we want to complete a basis of $\mathbb{R}^4$ and obtain the coordinates of the vector $v = (4, 1, 0, 1)$ in such a base.

6. Given the the basis of $\mathbb{R}^3$

$$\mathcal{B}_1 = \{(1,0,-2),(0,2,1),(1,1,-2)\},$$

obtain the matrix of the change of basis from $\mathcal{B}_1$ to the canonical basis $\mathcal{B}_c$ and from $\mathcal{B}_c$ to $\mathcal{B}_1$, where the canonical basis of $\mathbb{R}^3$ is

$$\mathcal{B}_c = \{(1,0,0),(0,1,0),(0,0,1)\}.$$

Given te vector $u = (1,-1,3)$. Obtain its coordinates with respect to the basis $\mathcal{B}_1$.

7. We consider the following basis of $\mathbb{R}^3$:

$$\mathcal{B}_1 = \{(0,2,2),(2,0,-1),(3,0,0)\} \quad \text{and} \quad \mathcal{B}_2 = \{(2,0,2),(1,2,0),(0,0,1)\}.$$

   (a) Obtain the matrix of the change of basis from $\mathcal{B}_1$ to $\mathcal{B}_2$.
   (b) Given the vector $(1,1,1)$ with coordinates with respect to $\mathcal{B}_1$, obtain its coordinates with respect to $\mathcal{B}_2$.

8. In the vectorial space $P_1(\mathbb{R})$ we consider the basis

$$\mathcal{B}_1 = \{t, 1\} \quad \text{and} \quad \mathcal{B}_2 = \{1+t, 1-t\}.$$

We want to:

   (a) Obtain the matrices of change of basis from $\mathcal{B}_1$ to $\mathcal{B}_2$, and the one form $\mathcal{B}_2$ to $\mathcal{B}_1$.
   (b) Given the polynomial $p(t) = t+3$, use the correct matrix of change of basis to obtain its coordinates with respect to the basis $\mathcal{B}_2$.

9. In the vectorial space $M_{2\times 2}(\mathbb{R})$ we consider the basis

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Find the matrix of change of basis from the canonical basis of $M_{2\times 2}(\mathbb{R})$ to $\mathcal{B}_1$, and obtain the coordinates of

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$$

with respect to $\mathcal{B}_1$.

10. In the vectorial space $M_{2\times 2}(\mathbb{R})$ we consider the canonical basis $\mathcal{B}_c$, and the basis

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

We want to:

(a) Find the matrix of change of basis from $\mathcal{B}_1$ to $\mathcal{B}_c$, and from $\mathcal{B}_c$ to $\mathcal{B}_1$.

(b) Given the matrix

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

obtain its coordinates with respect to the basis $\mathcal{B}_1$.

11. Given the following basis of $\mathbb{R}^4$:

$$\mathcal{B} = \{(1,0,2,1), (-1,1,0,0), (-1,0,2,1), (1,-1,0,1)\},$$

we want to obtain the matrix of the change of basis from $\mathcal{B}$ to the canonical basis of $\mathbb{R}^4$, the matrix of the change of basis from the canonical basis of $\mathbb{R}^4$ to $\mathcal{B}$, and the coordinates of the vector $(1,-2,0,1)$ with respect to the basis $\mathcal{B}$.

12. In the vectorial space $\mathbb{P}_3(\mathbb{R})$ we consider the canonical basis $\mathcal{B}_c = \{1, t, t^2, t^3\}$ and the basis $\mathcal{B}_1 = \{t^2 + t^3, t + t^2, t, 1\}$.

We want to:

(a) Obtain the matrix of change of basis from $\mathcal{B}_1$ to $\mathcal{B}_c$, and from $\mathcal{B}_c$ to $\mathcal{B}_1$.

(b) Given the polynomial $1 + t + t^2 + t^3$, use the right matrix of change of basis to obtain its coordinates in the basis $\mathcal{B}_1$.

13. In the vectorial space $\mathbb{P}_3(\mathbb{R})$ we consider the basis $\mathcal{B}_1 = \{t^3, t^2, t, 1\}$ and the basis $\mathcal{B}_1 = \{t^3, t + t^2, 1 + t, 1\}$.

We want to:

(a) Obtain the matrix of change of basis from $\mathcal{B}_1$ to $\mathcal{B}_2$, and from $\mathcal{B}_2$ to $\mathcal{B}_1$.

(b) Given the polynomial $t^3 + t^2 + t$, use the right matrix of change of basis to obtain its coordinates in the basis $\mathcal{B}_2$.