Practice exercises

Linear Mappings – Survey

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ the linear mapping defined by

$$f(x, y) = (x + y, 2x - y, 2x + 2y).$$

   (a) Obtain the coordinate matrix of f with respect to the canonical bases.

   (b) Obtain a basis, the dimension and a set of equations of the kernel of f and of the image space of f.

   (c) Obtain the coordinate matrix of f with respect to the basis $\mathcal{B}_1 = \{(1, 1), (-1, 2)\}$ of $\mathbb{R}^2$ and of the basis $\mathcal{B}_2 = \{(-1, 1, 0), (1, 0, 1), (0, 0, 1)\}$ of $\mathbb{R}^3$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

$$f(x, y, z) = (x + 2y - z, y + z, x + y - 2z).$$

   (a) Obtain the matrix of f with respect to the basis $\mathcal{B} = \{(0, 2, 1), (1, 0, 1), (-1, 0, 0)\}$.

   (b) Use this matrix to compute $f(-2, 2, 2)$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear mapping defined by:

$$f(1, 0, 1) = (1, 1, 1, 0)$$
$$f(-1, 2, 0) = (1, -3, 0, 1)$$
$$f(0, -1, -1) = (-1, 0, -1, 0).$$

   Obtain the matrix associated to f with respect to the canonical bases and $\mathrm{Ker}(f)$.

4. Let $f : \mathcal{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ be the linear mapping defined by:

$$f\left(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) = (x + y, z, -t).$$

   We want:

   (a) Prove that the application f is linear.

   (b) Obtain a basis, the dimension and equations of the $\mathrm{Ker}(f)$ and the image of f.

5. Let $f : \mathcal{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^4$ be the linear mapping defined by:

$$f\left(a_0 + a_1 t + a_2 t^2\right) = (a_0 + a_1 + a_2, a_0 - 2a_1, a_2, -a_0),$$

where E is the vector space of the polynomials of degree less or equal to 2 with real coefficients. Find the coordinate matrix of f with respect to the bases

$$\mathcal{B}_1 = \{1, t, t^2\},$$

and

$$\mathcal{B}_2 = \{(1,0,0,1), (0,1,0,0), (0,0,1,1), (1,1,1,0)\}.$$

Use such matrix to obtain $f(1+t)$.

6. Let E be the space of polynomials of degree at most 2 with real coefficients and let F be the 2-by-2 squared matrices with real entries. Let us consider the linear mapping $f : E \to F$ defined as follows:

$$f(a + bt + ct^2) = \begin{pmatrix} p(0) & p(1) \\ p(1) & p(0) \end{pmatrix}$$

   (a) Obtain the coordinate matrix of f with respect to the basis

   $$\mathcal{B}_1 = \{1, 1+t, 1+t+t^2\}$$

   of E and the canonical basis of F.

   (b) By using the matrix obtained in (a), compute $f(p)$ where $p = 3 + 2t + t^2$.

   (c) By using the matrix obtained in (a), compute a basis of the subpsace $\mathrm{Ker}(f)$.

7. Let E be the 2-by-2 squared matrices with real entries and let F be the space of polynomials of degree at most 2 with real coefficients. We consider the linear mapping $f : E \to F$ defined as follows:

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + (b+c)t + dt^2.$$

   (a) Obtain the coordinate matrix of f with respect to the basis

   $$\mathcal{B}_1 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

   of E and the canonical basis of F.

   (b) By using the matrix obtained in (a), compute $f(C)$, being $C = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$.

8. Let E be the space of 2-by-2 matrices with real entries. We consider the linear mapping $f : E \to E$ given by:

$$f(A) = A + A^T,$$

where A^T is the matrix transposed of A.

(a) Prove that f is a linear mapping.

(b) Obtain the coordinate matrix of f with respect to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

of E.

(c) By using the matrix obtained in (b), compute $f(C)$, being $C = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

9. Let E be the space of polynomials of degree at most 2 with real coefficients. We consider the linear mapping $f : E \to E$ given by:

$$f(p(t)) = p'(t),$$

(i.e. the image of a polynomial is its derivative).

(a) Obtain the coordinate matrix of f with respect to the basis $\mathcal{B} = \{1, 1+t, t^2\}$ of E.

(b) By using the matrix obtained in (a), compute $f(1 + 2t + 2t^2)$.

10. Let E be the space of polynomials of degree at most 2 with real coefficients and let F be the space of polynomials of degree at most 3 with real coefficients. We consider the linear mapping $f : E \to F$ defined as follows

$$f(p(t)) = \int_0^t p(s)\, ds$$

(i.e. the image of a polynomial is its definite integral between 0 and t: a primitive of the polynomial so that at 0 is equal to 0.)

(a) Obtain the coordinate matrix of f with respect to the basis $\mathcal{B} = \{1, 1+t, 1+t^2\}$ of E and the canonical basis of F.

(b) By using the matrix obtained in (a), compute $f(3 + t + t^2)$.