



11/01/2018	Cálculo I	Curso 2017-18	<b>Ex. Enero</b>
Apellidos:		Nombre:	Grupo:

— Primer parcial (1 hora) —

1. (5 puntos) Estudiar la convergencia en términos del parámetro  $\alpha \geq 0$ :

$$\sum_{n=1}^{\infty} \alpha^{n^2}.$$

**Solución:** Veamos para qué valores de  $\alpha$  se cumple la condición necesaria de convergencia:

$$\lim_{n \rightarrow \infty} \alpha^{n^2} = \begin{cases} +\infty \neq 0, & \text{si } \alpha > 1 \\ 1 \neq 0, & \text{si } \alpha = 1 \\ 0, & \text{si } 0 \leq \alpha < 1 \end{cases}$$

Por tanto solo puede converger para  $0 \leq \alpha < 1$ . Para este caso aplicaremos el criterio del cociente quedando

$$\frac{a_{n+1}}{a_n} = \frac{\alpha^{(n+1)^2}}{\alpha^{n^2}} = \alpha^{2n+1},$$

por tanto si  $0 \leq \alpha < 1$ , entonces

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \alpha^{2n+1} = 0 < 1,$$

por tanto para  $0 \leq \alpha < 1$  la serie es convergente; y para  $\alpha \geq 1$  es divergente.

2. (5 puntos) Hallar la expresión de la siguiente señal en el intervalo  $(-T/2, T/2)$  teniendo en cuenta que es una señal periódica de período fundamental  $T$ :

$$x(t) = \text{Impar}\{|\text{sen } t|\}, \quad \frac{7\pi}{2} < t < 4\pi.$$

**Solución:** La señal  $|\text{sen } t|$  tiene período  $\pi$  y el intervalo que nos dan tiene longitud  $L = \pi/2$  por tanto la señal tiene período  $T = \pi/2$ . Además en este caso

$$-\frac{T}{2} = -\frac{\pi}{4} = \frac{7\pi}{2} - kT \Rightarrow k = 7.5.$$

Por tanto

$$x(t) = \begin{cases} x(t+8T), & \text{si } -\frac{\pi}{4} < t < 0 \\ x(t+7T), & \text{si } 0 < t < \frac{\pi}{4} \end{cases}$$

Por tanto

$$\text{Impar}\{x(t)\} = \frac{x(t) - x(-t)}{2} = \begin{cases} \frac{x(t+8T) - x(-t+7T)}{2}, & \text{si } -\frac{\pi}{4} < t < 0 \\ \frac{x(t+7T) - x(-t+8T)}{2}, & \text{si } 0 < t < \frac{\pi}{4} \end{cases}$$

En el primer caso

$$x(t+8T) - x(-t+7T) = |\text{sen}(t+4\pi)| - |\text{sen}(-t+7\pi/2)| = |\text{sen}(t)| - |-\cos(t)|,$$

y dado que lo miramos en  $(-\pi/4, 0)$  entonces

$$x(t+8T) - x(-t+7T) = -\text{sen}(t) - \cos(t),$$

y por tanto  $x(t + 7T) - x(-t + 8T) = \cos(t) - \text{sen}(t)$ . Es decir,

$$\text{Impar}\{x(t)\} = \frac{x(t) - x(-t)}{2} = \begin{cases} -\frac{\cos(t) + \text{sen}(t)}{2}, & \text{si } -\frac{\pi}{4} < t < 0 \\ \frac{\cos(t) - \text{sen}(t)}{2}, & \text{si } 0 < t < \frac{\pi}{4} \end{cases}$$

▬ Segundo parcial (1 hora) ▬

1. (5 puntos) Calcular la integral:

$$\int \frac{\sec^4(x)}{8 + \tan^3(x)} dx.$$

**Solución:** Aplicamos el cambio  $\tan x = z$ , por tanto  $(1 + \tan^2(x)) dx = dz$  así

$$\int \frac{\sec^4(x)}{8 + \tan^3(x)} dx = \int \frac{(1 + \tan^2(x))^2}{8 + \tan^3(x)} dx = \int \frac{1 + z^2}{8 + z^3} dz.$$

Dado que  $z^3 + 8 = (z + 2)(z^2 - 2z + 4)$  y  $z^2 - 2z + 4 = (z - 1)^2 + (\sqrt{3})^2$  entonces

$$\int \frac{1 + z^2}{8 + z^3} dz = \int \frac{A}{z + 2} dz + \int \frac{M(z - 1) + N}{(z - 1)^2 + (\sqrt{3})^2} dz.$$

Es decir

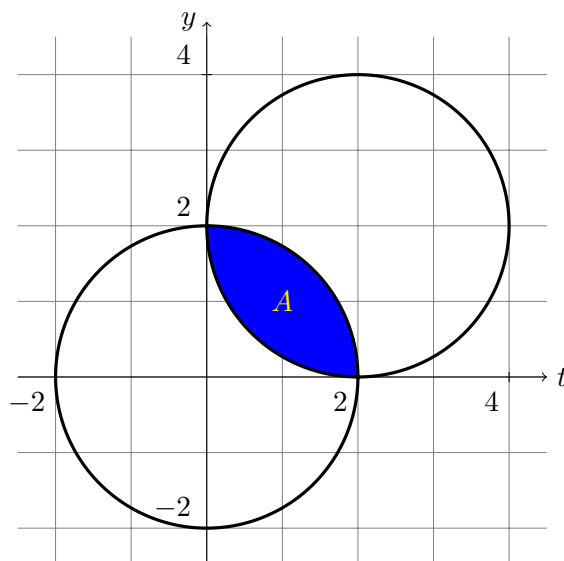
$$\int \frac{1 + z^2}{8 + z^3} dz = A \ln|z + 2| + \frac{M}{2} \ln|z^2 - 2z + 4| + \frac{N}{\sqrt{3}} \arctan\left(\frac{z - 1}{\sqrt{3}}\right) + C$$

En este caso  $A = 5/12$ ,  $M = 7/12$  y  $N = 1/4$ .

2. (5 puntos) Calcular el área encerrada entre las circunferencias:

$$x^2 + y^2 = 4, \quad (x - 2)^2 + (y - 2)^2 = 4.$$

**Solución:** Si hacemos un esbozo de las circunferencias se tiene



En este caso,

$$A = \int_0^2 \left( \sqrt{4-x^2} - (2 - \sqrt{4-(x-2)^2}) \right) dx = \int_0^2 \sqrt{4-x^2} dx - \int_0^2 (2 - \sqrt{4-(x-2)^2}) dx$$

En el primer caso (CV:  $x = 2\text{sen}(z)$ )

$$\int_0^2 \sqrt{4-x^2} dx \stackrel{CV}{=} 4 \int_0^{\frac{\pi}{2}} \cos^2(z) dz = 2 \int_0^{\frac{\pi}{2}} (1 + \cos(2z)) dz = \pi,$$

y en el segundo caso (CV:  $x - 2 = 2\text{sen}(z)$ )

$$\int_0^2 (2 - \sqrt{4-(x-2)^2}) dx = 4 - \int_0^2 \sqrt{4-(x-2)^2} dx \stackrel{CV}{=} 4 \left( 1 - \int_0^{\frac{\pi}{2}} \cos^2(z) dz \right) = 4 - \pi.$$

Es decir, que  $A = 2\pi - 4$ .

▬ Tercer parcial (2 horas) ▬

1. (3 puntos) Calcular la convolución  $y(t) = x(t) * h(t)$ , con

$$x(t) = e^{-t} \mathbf{u}(t+2), \quad h(t) = e^t \mathbf{u}(-3-t).$$

2. (3'5 puntos) Resolver el siguiente problema de Cauchy empleando la transformada unilateral de Laplace:

$$\begin{cases} x''(t) - x(t) = t \mathbf{u}(t-2) \\ x(0) = 2, \quad x'(0) = 0 \end{cases}$$

**Solución:** Si llamamos  $X(s) = \mathcal{L}\{x(t)\}(s)$ , entonces aplicando la propiedad

$$\mathcal{L}\{x''(t)\}(s) = s^2 X(s) - sx(0^+) - x'(0^+) = s^2 X(s) - 2s,$$

así la ecuación diferencial al aplicarle la transformada de Laplace resulta

$$(s^2 X(s) - 2s) - X(s) = \mathcal{L}\{t \mathbf{u}(t-2)\}(s)$$

y queda tras simplificar

$$(s^2 - 1)X(s) = 2s + \mathcal{L}\{t \mathbf{u}(t-2)\}(s)$$

Ahora aplicamos la propiedad

$$\mathcal{L}\{h(t-t_0)\mathbf{u}(t-t_0)\}(s) = e^{-st_0} \mathcal{L}\{h(t)\}(s).$$

Dado que este caso  $t_0 = 2$  y  $h(t-2) = t$ , entonces  $h(t) = t+2$ , luego

$$(s-1)(s+1)X(s) = 2s + e^{-2s} \left( \frac{1}{s^2} + \frac{2}{s} \right),$$

es decir,

$$(*) \quad X(s) = \frac{2s}{(s-1)(s+1)} + e^{-2s} \frac{2s+1}{(s-1)(s+1)s^2} = F(s) + G(s)e^{-2s}.$$

En este caso, dado que

$$F(s) = \frac{2s}{(s-1)(s+1)} = \frac{1}{s-1} + \frac{1}{s+1} = \mathcal{L}\{e^t + e^{-t}\}(s) = \mathcal{L}\{f(t)\}(s).$$

y que

$$\begin{aligned} G(s) &= \frac{2s+1}{(s-1)(s+1)s^2} = -\frac{1}{s^2} - \frac{2}{s} + \frac{3}{2(s-1)} + \frac{1}{2(s+1)} \\ &= \mathcal{L}\{-t - 2 + 3/2e^t + 1/2e^{-t}\}(s) = \mathcal{L}\{g(t)\}(s). \end{aligned}$$

Aplicando de nuevo la propiedad anterior, se tiene que

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\{f(t)\}(s) + e^{-2s}\mathcal{L}\{g(t)\}(s),$$

Es decir,

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\{e^t + e^{-t} + g(t-2)u(t-2)\}(s).$$

Luego la solución es  $x(t) = e^t + e^{-t} + \frac{1}{2}(-2t + 3e^{t-2} + e^{-t+2})u(t-2)$ .

**3. (3'5 puntos)** Calcular el desarrollo en serie de Fourier trigonométrica de la señal:

$$x(t) = t u(t), \quad -\pi < t < \pi.$$

**Solución:** Tenemos que

$$\mathcal{F}\{x(t)\} = \frac{a_0}{2} + \sum_{k \geq 1} a_k \cos(kw_0 t) + \sum_{k \geq 1} b_k \sin(kw_0 t),$$

con  $T = 2\pi$  y  $w_0 = \frac{2\pi}{T} = 1$ . De modo que

$$\begin{aligned} a_0 &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{1}{\pi} \int_0^{\pi} t dt = \frac{\pi}{2}, \\ a_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(kt) dt = \frac{1}{\pi} \int_0^{\pi} t \cos(kt) dt = \frac{(-1)^k - 1}{\pi k^2}, \\ b_k &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(kt) dt = \frac{1}{\pi} \int_0^{\pi} t \sin(kt) dt = \frac{(-1)^{k+1}}{k}, \end{aligned}$$

donde las últimas dos integrales se resuelven integrando por partes.



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— First partial exam (1 hour) —

1. (5 points) Study the convergence of the following series in terms of the parameter  $\alpha \geq 0$ :

$$\sum_{n=1}^{\infty} \alpha^{n^2}.$$

**Solution:** We check the behavior of the series for the values of  $\alpha$  in terms of the sufficient condition for convergence:

$$\lim_{n \rightarrow \infty} \alpha^{n^2} = \begin{cases} +\infty \neq 0, & \text{if } \alpha > 1 \\ 1 \neq 0, & \text{if } \alpha = 1 \\ 0, & \text{if } 0 \leq \alpha < 1 \end{cases}$$

Then, it can only converge for  $0 \leq \alpha < 1$ . In this case, we apply the quotient criteria to get

$$\frac{a_{n+1}}{a_n} = \frac{\alpha^{(n+1)^2}}{\alpha^{n^2}} = \alpha^{2n+1},$$

then, if  $0 \leq \alpha < 1$ , we have

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \alpha^{2n+1} = 0 < 1,$$

and for  $0 \leq \alpha < 1$  the series is convergent; for  $\alpha \geq 1$  it is divergent.

2. (5 points) Write the following signal in the interval  $(-T/2, T/2)$ , taking into account that it is a periodic signal of period  $T$ :

$$x(t) = \text{Odd}\{|\sin(t)|\}, \quad \frac{7\pi}{2} < t < 4\pi.$$

**Solution:** The signal  $|\sin(t)|$  has period  $\pi$  and the interval given has length  $L = \pi/2$ . Then, the signal has period given by  $T = \pi/2$ . Moreover, in this case

$$-\frac{T}{2} = -\frac{\pi}{4} = \frac{7\pi}{2} - kT \quad \Rightarrow \quad k = 7.5.$$

Then,

$$x(t) = \begin{cases} x(t+8T), & \text{if } -\frac{\pi}{4} < t < 0 \\ x(t+7T), & \text{if } 0 < t < \frac{\pi}{4} \end{cases}$$

Then

$$\text{Odd}\{x(t)\} = \frac{x(t) + x(-t)}{2} = \begin{cases} \frac{x(t+8T) - x(-t-7T)}{2}, & \text{if } -\frac{\pi}{4} < t < 0 \\ \frac{x(t+7T) - x(-t-8T)}{2}, & \text{if } 0 < t < \frac{\pi}{4} \end{cases}$$

In the first case

$$x(t+8T) - x(-t-7T) = |\sin(t+4\pi)| - |\sin(-t-7\pi/2)| = |\sin(t)| - |-\cos(t)|,$$

and in the interval  $(-\pi/4, 0)$  we get

$$x(t+8T) - x(-t-7T) = -\sin(t) - \cos(t),$$

so  $x(t + 7T) - x(-t - 8T) = \cos(t) - \sin(t)$ . This means that

$$\text{Odd}\{x(t)\} = \frac{x(t) + x(-t)}{2} = \begin{cases} -\frac{\cos(t)+\sin(t)}{2}, & \text{if } -\frac{\pi}{4} < t < 0 \\ \frac{\cos(t)-\sin(t)}{2}, & \text{if } 0 < t < \frac{\pi}{4} \end{cases}$$

———— Second partial exam (1 hour) ————

1. (5 points) Find:

$$\int \frac{\sec^4(x)}{8 + \tan^3(x)} dx.$$

**Solution:** We apply the change of variable  $\tan x = z$  to get  $(1 + \tan^2(x)) dx = dz$  and

$$\int \frac{\sec^4(x)}{8 + \tan^3(x)} dx = \int \frac{(1 + \tan^2(x))^2}{8 + \tan^3(x)} dx = \int \frac{1 + z^2}{8 + z^3} dz.$$

As  $z^3 + 8 = (z + 2)(z^2 - 2z + 4)$  and  $z^2 - 2z + 4 = (z - 1)^2 + (\sqrt{3})^2$  then

$$\int \frac{1 + z^2}{8 + z^3} dz = \int \frac{A}{z + 2} dz + \int \frac{M(z - 1) + N}{(z - 1)^2 + (\sqrt{3})^2} dz.$$

Then,

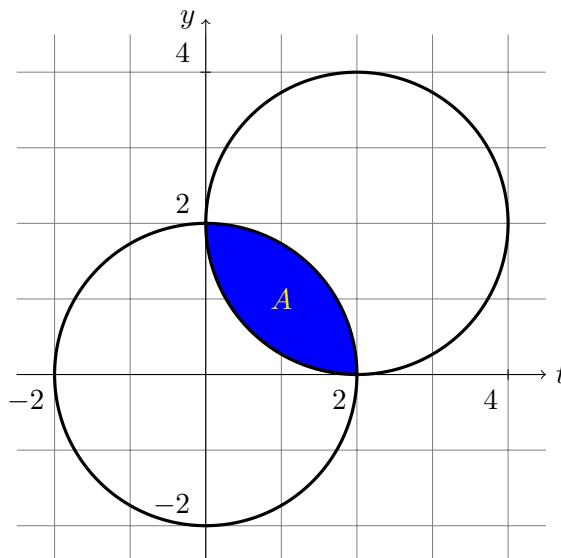
$$\int \frac{1 + z^2}{8 + z^3} dz = A \ln |z + 2| + \frac{M}{2} \ln |z^2 - 2z + 4| + \frac{N}{\sqrt{3}} \arctan \left( \frac{z - 1}{\sqrt{3}} \right) + C$$

In this case  $A = 5/12$ ,  $M = 7/12$  y  $N = 1/4$ .

2. (5 points) Find the area inside the circles:

$$x^2 + y^2 = 4, \quad (x - 2)^2 + (y - 2)^2 = 4.$$

**Solution:** A sketch of the picture is the following:



In this case,

$$A = \int_0^2 \left( \sqrt{4-x^2} - (2 - \sqrt{4-(x-2)^2}) \right) dx = \int_0^2 \sqrt{4-x^2} dx - \int_0^2 (2 - \sqrt{4-(x-2)^2}) dx$$

In the first case (CV:  $x = 2 \sin(z)$ )

$$\int_0^2 \sqrt{4-x^2} dx \stackrel{CV}{=} 4 \int_0^{\frac{\pi}{2}} \cos^2(z) dz = 2 \int_0^{\frac{\pi}{2}} (1 + \cos(2z)) dz = \pi,$$

and in the second (CV:  $x - 2 = 2 \sin(z)$ )

$$\int_0^2 (2 - \sqrt{4-(x-2)^2}) dx = 4 - \int_0^2 \sqrt{4-(x-2)^2} dx \stackrel{CV}{=} 4(1 - \int_0^{\frac{\pi}{2}} \cos^2(z) dz) = 4 - \pi.$$

Es decir, que  $A = 2\pi - 4$ .

▬ Third partial exam (2 hours) ▬

1. (3 points) Find the convolution  $y(t) = x(t) * h(t)$ , where

$$x(t) = e^{-t} \mathbf{u}(t+2), \quad h(t) = e^t \mathbf{u}(-3-t).$$

2. (3'5 points) Solve the following Cauchy problem by means of unilateral Laplace transform:

$$\begin{cases} x''(t) - x(t) = t \mathbf{u}(t-2) \\ x(0) = 2, \quad x'(0) = 0 \end{cases}$$

**Solution:** Let  $X(s) = \mathcal{L}\{x(t)\}(s)$ , then we apply

$$\mathcal{L}\{x''(t)\}(s) = s^2 X(s) - sx(0^+) - x'(0^+) = s^2 X(s) - 2s,$$

to transform the differential equation, after Laplace transform, into

$$(s^2 X(s) - 2s) - X(s) = \mathcal{L}\{t \mathbf{u}(t-2)\}(s)$$

and we obtain

$$(s^2 - 1)X(s) = 2s + \mathcal{L}\{t \mathbf{u}(t-2)\}(s)$$

We now apply

$$\mathcal{L}\{h(t-t_0)\mathbf{u}(t-t_0)\}(s) = e^{-st_0} \mathcal{L}\{h(t)\}(s).$$

In this case  $t_0 = 2$  and  $h(t-2) = t$ , then  $h(t) = t+2$ , and then

$$(s-1)(s+1)X(s) = 2s + e^{-2s} \left( \frac{1}{s^2} + \frac{2}{s} \right),$$

it is to say

$$(*) \quad X(s) = \frac{2s}{(s-1)(s+1)} + e^{-2s} \frac{2s+1}{(s-1)(s+1)s^2} = F(s) + G(s)e^{-2s}.$$

As we have

$$F(s) = \frac{2s}{(s-1)(s+1)} = \frac{1}{s-1} - \frac{1}{s+1} = \mathcal{L}\{e^t - e^{-t}\}(s) = \mathcal{L}\{f(t)\}(s).$$

and

$$\begin{aligned} G(s) &= \frac{2s+1}{(s-1)(s+1)s^2} = -\frac{1}{s^2} - \frac{2}{s} + \frac{3}{2(s-1)} + \frac{1}{2(s+1)} \\ &= \mathcal{L}\{-t - 2 + 3/2e^t + 1/2e^{-t}\}(s) = \mathcal{L}\{g(t)\}(s). \end{aligned}$$

We apply the last property to get

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\{f(t)\}(s) + e^{-2s}\mathcal{L}\{g(t)\}(s),$$

It is to say,

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\{e^t - e^{-t} + g(t-2)\mathbf{u}(t-2)\}(s).$$

Then, the solution is  $x(t) = e^t - e^{-t} + \frac{1}{2}(-2t + 3e^{t-2} + e^{-t+2})\mathbf{u}(t-2)$ .

**3. (3'5 points)** Find the Fourier series in its trigonometric form of the following periodic signal:

$$x(t) = t\mathbf{u}(t), \quad -\pi < t < \pi.$$

**Solution:** We have

$$\mathcal{F}\{x(t)\} = \frac{a_0}{2} + \sum_{k \geq 1} a_k \cos(kw_0t) + \sum_{k \geq 1} b_k \sin(kw_0t),$$

with  $T = 2\pi$  and  $w_0 = \frac{2\pi}{T} = 1$ . We have

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{1}{\pi} \int_0^{\pi} t dt \frac{\pi}{2},$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(kt) dt = \frac{1}{\pi} \int_0^{\pi} t \cos(kt) dt = \frac{(-1)^k - 1}{\pi k^2},$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(kt) dt = \frac{1}{\pi} \int_0^{\pi} t \sin(kt) dt = \frac{(-1)^{k+1}}{k},$$

where the last two integrals are solved by integration by parts.