Dpto. Física y Matemáticas

| 11/01/2018 | Cálculo I | Curso 2017-18 | **Ex. Enero** |
|---|---|---|---|
| Apellidos: | | Nombre: | Grupo: |

**Primer parcial (1 hora)**

**1. (5 puntos)** Estudiar la convergencia en términos del parámetro $\alpha \geq 0$:

$$\sum_{n=1}^{\infty} \alpha^{n^2}.$$

**Solución:** Veamos para qué valores de $\alpha$ se cumple la condición necesaria de convergencia:

$$\lim_{n\to\infty} \alpha^{n^2} = \begin{cases} +\infty \neq 0, & si\ \alpha > 1 \\ 1 \neq 0 &, si\ \alpha = 1 \\ 0 &, si\ 0 \leq \alpha < 1 \end{cases}$$

Por tanto solo puede converger para $0 \leq \alpha < 1$. Para este caso aplicaremos el criterio del cociente quedando

$$\frac{a_{n+1}}{a_n} = \frac{\alpha^{(n+1)^2}}{\alpha^{n^2}} = \alpha^{2n+1},$$

por tanto si $0 \leq \alpha < 1$, entonces

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \alpha^{2n+1} = 0 < 1,$$

por tanto para $0 \leq \alpha < 1$ la serie es convergente; y para $\alpha \geq 1$ es divergente.

**2. (5 puntos)** Hallar la expresión de la siguiente señal en el intervalo $(-T/2, T/2)$ teniendo en cuenta que es una señal periódia de período fundamental $T$:

$$x(t) = \text{Impar}\{|\text{sen}\, t|\}, \qquad \frac{7\pi}{2} < t < 4\pi.$$

**Solución:** La señal $|\text{sen}\, t|$ tiene período $\pi$ y el intervalo que nos dan tiene longitud $L = \pi/2$ por tanto la señal tiene período $T = \pi/2$. Además en este caso

$$-\frac{T}{2} = -\frac{\pi}{4} = \frac{7\pi}{2} - kT \quad \Rightarrow \quad k = 7'5.$$

Por tanto

$$x(t) = \begin{cases} x(t+8T), & si\ -\frac{\pi}{4} < t < 0 \\ x(t+7T), & si\ 0 < t < \frac{\pi}{4} \end{cases}$$

Por tanto

$$\text{Impar}\{x(t)\} = \frac{x(t) - x(-t)}{2} = \begin{cases} \frac{x(t+8T) - x(-t+7T)}{2}, & si\ -\frac{\pi}{4} < t < 0 \\ \frac{x(t+7T) - x(-t+8T)}{2}, & si\ 0 < t < \frac{\pi}{4} \end{cases}$$

En el primer caso

$$x(t+8T) - x(-t+7T) = |\text{sen}(t+4\pi)| - |\text{sen}(-t+7\pi/2)| = |\text{sen}(t)| - |-\cos(t)|,$$

y dado que lo miramos en $(-\pi/4, 0)$ entonces

$$x(t+8T) - x(-t+7T) = -\text{sen}(t) - \cos(t),$$

y por tanto $x(t+7T)-x(-t+8T)=\cos(t)-\operatorname{sen}(t)$. Es decir,

$$\operatorname{Impar}\{x(t)\}=\frac{x(t)-x(-t)}{2}=\begin{cases} -\dfrac{\cos(t)+\operatorname{sen}(t)}{2}, & si\ -\dfrac{\pi}{4}<t<0 \\ \dfrac{\cos(t)-\operatorname{sen}(t)}{2}, & si\ 0<t<\dfrac{\pi}{4} \end{cases}$$

## Segundo parcial (1 hora)

**1. (5 puntos)** Calcular la integral:

$$\int \frac{\sec^4(x)}{8+\tan^3(x)}\,dx.$$

**Solución:** Aplicamos el cambio $\tan x = z$, por tanto $(1+\tan^2(x))\,dx = dz$ así

$$\int \frac{\sec^4(x)}{8+\tan^3(x)}\,dx=\int \frac{(1+\tan^2(x))^2}{8+\tan^3(x)}\,dx=\int \frac{1+z^2}{8+z^3}\,dz.$$

Dado que $z^3+8=(z+2)(z^2-2z+4)$ y $z^2-2x+4=(z-1)^2+(\sqrt{3})^2$ entones

$$\int \frac{1+z^2}{8+z^3}\,dz=\int \frac{A}{z+2}\,dz+\int \frac{M(z-1)+N}{(z-1)^2+(\sqrt{3})^2}\,dz.$$

Es decir

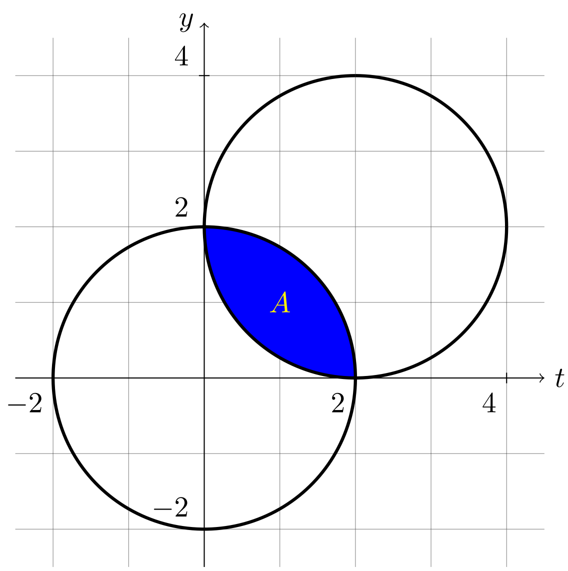
$$\int \frac{1+z^2}{8+z^3}\,dz=A\ln|z+2|+\frac{M}{2}\ln|z^2-2x+4|+\frac{N}{\sqrt{3}}\arctan\left(\frac{z-1}{\sqrt{3}}\right)+C$$

En este caso $A=5/12$, $M=7/12$ y $N=1/4$.

**2. (5 puntos)** Calcular el área encerrada entre las circunferencias:

$$x^2+y^2=4, \qquad (x-2)^2+(y-2)^2=4.$$

**Solución:** Si hacemos un esbozo de las circunferencias se tiene



En este caso,

$$A=\int_0^2\left(\sqrt{4-x^2}-(2-\sqrt{4-(x-2)^2})\right)dx=\int_0^2\sqrt{4-x^2}dx-\int_0^2(2-\sqrt{4-(x-2)^2})dx$$

En el primer caso (CV: $x=2\mathrm{sen}(z)$)

$$\int_0^2\sqrt{4-x^2}\,dx\overset{CV}{=}4\int_0^{\frac{\pi}{2}}\cos^2(z)\,dz=2\int_0^{\frac{\pi}{2}}(1+\cos(2z))\,dz=\pi,$$

y en el segundo caso (CV: $x-2=2\mathrm{sen}(z)$)

$$\int_0^2(2-\sqrt{4-(x-2)^2})dx=4-\int_0^2\sqrt{4-(x-2)^2}dx\overset{CV}{=}4(1-\int_0^{\frac{\pi}{2}}\cos^2(z)\,dz)=4-\pi.$$

Es decir, que $A=2\pi-4$.

**Tercer parcial (2 horas)**

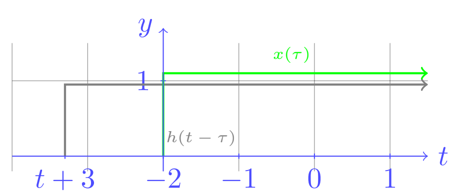
**1. (3 puntos)** Calcular la convolución $y(t)=x(t)*h(t)$, con

$$x(t)=e^{-t}\,\mathtt{u}(t+2),\qquad h(t)=e^{t}\,\mathtt{u}(-3-t).$$

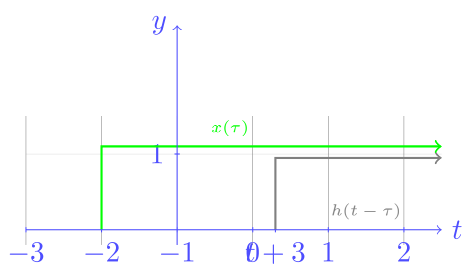
**Solución:** Si $x(t)=e^{-t}\,\mathtt{u}(t+2)$ entonces $x(\tau)=e^{-\tau}\,\mathtt{u}(\tau+2)$ (que contiene un salto a la derecha empezando en -2), y si $h(t)=e^{t}\,\mathtt{u}(-3-t)$ entonces $h(t-\tau)=e^{t-\tau}\,\mathtt{u}(-3-t+\tau)$ (hay que verla como función en la variable $\tau$, y no en $t$) que contiene un salto a la derecha empezando en $t+3$. Por tanto

$$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau=\int_{-\infty}^{\infty}e^{-\tau}\,\mathtt{u}(\tau+2)e^{t-\tau}\,\mathtt{u}(-3-t+\tau)\,d\tau.$$

Dado que tenemos dos saltos y uno de ellos depende del parámetro $t$, tenemos que distinguir dos casos:
(a) que $t+3\le-2$, y (b) que $t+3>-2$. En el primer caso




y en el segundo



Teniendo en cuenta los esquemas, es claro que en el caso (a) ($t \le -5$) por tanto

$$x(t) * h(t) = \int_{-2}^{\infty} e^{t-2\tau}\, d\tau = \left( \frac{e^{t-2\tau}}{-2} \right|_{\tau=-2}^{\tau=\infty} = 0 - \left( \frac{e^{t+4}}{-2} \right) = \frac{e^{t+4}}{2}.$$

En el segundo caso (b) ($t > -5$), vemos que

$$x(t) * h(t) = \int_{t+3}^{\infty} e^{t-2\tau}\, d\tau = \left( \frac{e^{t-2\tau}}{-2} \right|_{\tau=t+3}^{\tau=\infty} = 0 - \left( \frac{e^{-t-6}}{-2} \right) = \frac{e^{-t-6}}{2}.$$

Así, la solución puede escribirse como una función definida a trozos de la forma, que a su vez puede expresarse como suma de saltos a la derecha:

$$x(t) * h(t) = \begin{cases} \dfrac{e^{t+4}}{2}, & si\ t \le -5 \\ \dfrac{e^{-t-6}}{2}, & si\ t > -5 \end{cases} = \frac{e^{t+4}}{2} + \frac{1}{2}(e^{-t-6} - e^{t+4})\, \mathtt{u}(t+5).$$

**2. (3'5 puntos)** Resolver el siguiente problema de Cauchy empleando la transformada unilateral de Laplace:

$$\begin{cases} x''(t) - x(t) = t\, \mathtt{u}(t-2) \\ x(0) = 2,\ x'(0) = 0 \end{cases}$$

**Solución:** Si llamamos $X(s) = \mathcal{L}\{x(t)\}(s)$, entonces aplicando la propiedad

$$\mathcal{L}\{x''(t)\}(s) = s^2 X(s) - s x(0^+) - x'(0^+) = s^2 X(s) - 2s,$$

así la ecuación diferencial al aplicarle la transformada de Laplace resulta

$$\left(s^2 X(s) - 2s\right) - X(s) = \mathcal{L}\{t\, \mathtt{u}(t-2)\}(s)$$

y queda tras simplificar

$$(s^2 - 1)X(s) = 2s + \mathcal{L}\{t\, \mathtt{u}(t-2)\}(s)$$

Ahora aplicamos la propiedad

$$\mathcal{L}\{h(t-t_0)\mathtt{u}(t-t_0)\}(s) = e^{-st_0}\mathcal{L}\{h(t)\}(s).$$

Dado que este caso $t_0 = 2$ y $h(t-2) = t$, entonces $h(t) = t + 2$, luego

$$(s-1)(s+1)X(s) = 2s + e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right),$$

es decir,

$$(*) \quad X(s) = \frac{2s}{(s-1)(s+1)} + e^{-2s}\frac{2s+1}{(s-1)(s+1)s^2} = F(s) + G(s)e^{-2s}.$$

En este caso, dado que

$$F(s) = \frac{2s}{(s-1)(s+1)} = \frac{1}{s-1} + \frac{1}{s+1} = \mathcal{L}\left\{e^t + e^{-t}\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s).$$

y que

$$G(s) = \frac{2s+1}{(s-1)(s+1)s^2} = -\frac{1}{s^2} - \frac{2}{s} + \frac{3}{2(s-1)} + \frac{1}{2(s+1)}$$
$$= \mathcal{L}\left\{-t - 2 + 3/2e^t + 1/2e^{-t}\right\}(s) = \mathcal{L}\{g(t)\}(s).$$

Aplicando de nuevo la propiedad anterior, se tiene que

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\left\{f(t)\right\}(s) + e^{-2s}\mathcal{L}\{g(t)\}(s),$$

Es decir,

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\left\{e^t + e^{-t} + g(t-2)\mathtt{u}(t-2)\right\}(s).$$

Luego la solución es $\boxed{x(t) = e^t + e^{-t} + \frac{1}{2}\left(-2t + 3e^{t-2} + e^{-t+2}\right)\mathtt{u}(t-2)}$.

**3. (3'5 puntos)** Calcular el desarrollo en serie de Fourier trigonométrica de la señal:

$$x(t) = t\,\mathtt{u}(t), \quad -\pi < t < \pi.$$

**Solución:** Tenemos que

$$\mathcal{F}\{x(t)\} = \frac{a_0}{2} + \sum_{k\geq 1} a_k \cos(kw_0 t) + \sum_{k\geq 1} b_k \sin(kw_0 t),$$

con $T = 2\pi$ y $w_0 = \frac{2\pi}{T} = 1$. De modo que

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{\pi}\int_0^{\pi} tdt\frac{\pi}{2},$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(kt)dt = \frac{1}{\pi}\int_0^{\pi} t\cos(kt)dt = \frac{(-1)^k - 1}{\pi k^2},$$
$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\text{sen}\,(kt)dt = \frac{1}{\pi}\int_0^{\pi} t\text{sen}\,(kt)dt = \frac{(-1)^{k+1}}{k},$$

donde las últimas dos integrales se resuelven integrando por partes.

**First partial exam (1 hour)**

**1. (5 points)** Study the convergence of the following series in terms of the parameter $\alpha \geq 0$:

$$\sum_{n=1}^{\infty} \alpha^{n^2}.$$

**Solution:** We check the behavior of the series for the values of $\alpha$ in terms of the sufficient condition for convergence:

$$\lim_{n\to\infty} \alpha^{n^2} = \begin{cases} +\infty \neq 0, & if\ \alpha > 1 \\ 1 \neq 0 & , if\ \alpha = 1 \\ 0 & , if\ 0 \leq \alpha < 1 \end{cases}$$

Then, it can only converge for $0 \leq \alpha < 1$. In this case, we apply the quotient criteria to get

$$\frac{a_{n+1}}{a_n} = \frac{\alpha^{(n+1)^2}}{\alpha^{n^2}} = \alpha^{2n+1},$$

then, if $0 \leq \alpha < 1$, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \alpha^{2n+1} = 0 < 1,$$

and for $0 \leq \alpha < 1$ the series is convergent; for $\alpha \geq 1$ it is divergent.

**2. (5 points)** Write the following signal in the interval $(-T/2, T/2)$, taking into account that it is a periodic signal of period $T$:

$$x(t) = \text{Odd}\{|\sin(t)|\}, \qquad \frac{7\pi}{2} < t < 4\pi.$$

**Solution:** The signal $|\sin(t)|$ has period $\pi$ and the interval given has lenght $L = \pi/2$. Then, the signal has period given by $T = \pi/2$. Moreover, in this case

$$-\frac{T}{2} = -\frac{\pi}{4} = \frac{7\pi}{2} - kT \quad \Rightarrow \quad k = 7'5.$$

Then,

$$x(t) = \begin{cases} x(t+8T), & if\ -\frac{\pi}{4} < t < 0 \\ x(t+7T), & if\ 0 < t < \frac{\pi}{4} \end{cases}$$

Then

$$\text{Odd}\{x(t)\} = \frac{x(t) + x(-t)}{2} = \begin{cases} \frac{x(t+8T) - x(-t-7T)}{2}, & if\ -\frac{\pi}{4} < t < 0 \\ \frac{x(t+7T) - x(-t-8T)}{2}, & if\ 0 < t < \frac{\pi}{4} \end{cases}$$

In the first case

$$x(t+8T) - x(-t-7T) = |\sin(t+4\pi)| - |\sin(-t-7\pi/2)| = |\sin(t)| - |-\cos(t)|,$$

and in the interval $(-\pi/4, 0)$ we get

$$x(t+8T) - x(-t-7T) = -\sin(t) - \cos(t),$$

so $x(t+7T)-x(-t-8T)=\cos(t)-\sin(t)$. This means that

$$\mathrm{Odd}\{x(t)\}=\frac{x(t)+x(-t)}{2}=\begin{cases} -\frac{\cos(t)+\sin(t)}{2}, & if\ -\frac{\pi}{4}<t<0 \\ \frac{\cos(t)-\sin(t)}{2}, & if\ 0<t<\frac{\pi}{4} \end{cases}$$

`Second partial exam (1 hour)`

**1. (5 points)** Find:

$$\int \frac{\sec^4(x)}{8+\tan^3(x)}\,dx.$$

**Solution:** We apply the change of variable $\tan x = z$ to get $(1+\tan^2(x))\,dx = dz$ and

$$\int \frac{\sec^4(x)}{8+\tan^3(x)}\,dx = \int \frac{(1+\tan^2(x))^2}{8+\tan^3(x)}\,dx = \int \frac{1+z^2}{8+z^3}\,dz.$$

As $z^3+8=(z+2)(z^2-2z+4)$ and $z^2-2x+4=(z-1)^2+(\sqrt{3})^2$ then

$$\int \frac{1+z^2}{8+z^3}\,dz = \int \frac{A}{z+2}\,dz + \int \frac{M(z-1)+N}{(z-1)^2+(\sqrt{3})^2}\,dz.$$

Then,

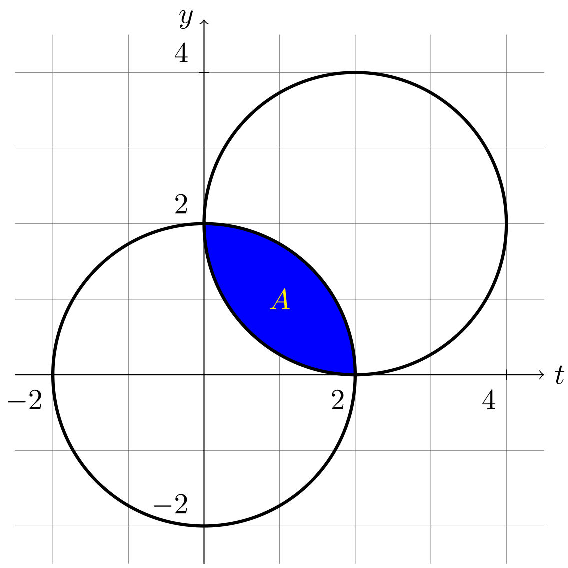
$$\int \frac{1+z^2}{8+z^3}\,dz = A\ln|z+2| + \frac{M}{2}\ln|z^2-2x+4| + \frac{N}{\sqrt{3}}\arctan\left(\frac{z-1}{\sqrt{3}}\right)+C$$

In this case $A=5/12$, $M=7/12$ y $N=1/4$.

**2. (5 points)** Find the area inside the circles:

$$x^2+y^2=4, \qquad (x-2)^2+(y-2)^2=4.$$

**Solution:** A scketch of the picture is the following:



In this case,

$$A = \int_0^2 \left(\sqrt{4-x^2} - (2-\sqrt{4-(x-2)^2})\right) dx = \int_0^2 \sqrt{4-x^2}dx - \int_0^2 (2-\sqrt{4-(x-2)^2})dx$$

In the first case (CV: $x = 2\sin(z)$)

$$\int_0^2 \sqrt{4-x^2}\,dx \overset{CV}{=} 4\int_0^{\frac{\pi}{2}} \cos^2(z)\,dz = 2\int_0^{\frac{\pi}{2}} (1+\cos(2z))\,dz = \pi,$$

and in the second (CV: $x-2 = 2\sin(z)$)

$$\int_0^2 (2-\sqrt{4-(x-2)^2})dx = 4 - \int_0^2 \sqrt{4-(x-2)^2}dx \overset{CV}{=} 4(1 - \int_0^{\frac{\pi}{2}} \cos^2(z)\,dz) = 4-\pi.$$

Es decir, que $A = 2\pi - 4$.

## Third partial exam (2 hours)

**1. (3 points)** Find the convolution $y(t) = x(t) * h(t)$, where

$$x(t) = e^{-t}\,\mathtt{u}(t+2), \qquad h(t) = e^{t}\,\mathtt{u}(-3-t).$$

**2. (3'5 points)** Solve the following Cauchy problem by means of unilateral Laplace transform:

$$\begin{cases} x''(t) - x(t) = t\,\mathtt{u}(t-2) \\ \quad x(0) = 2,\ x'(0) = 0 \end{cases}$$

**Solution:** Let $X(s) = \mathcal{L}\{x(t)\}(s)$, then we apply

$$\mathcal{L}\{x''(t)\}(s) = s^2X(s) - sx(0^+) - x'(0^+) = s^2X(s) - 2s,$$

to transform the differential equation, after Laplace transform, into

$$\left(s^2X(s) - 2s\right) - X(s) = \mathcal{L}\{t\,\mathtt{u}(t-2)\}(s)$$

and we obtain

$$(s^2-1)X(s) = 2s + \mathcal{L}\{t\,\mathtt{u}(t-2)\}(s)$$

We now apply

$$\mathcal{L}\{h(t-t_0)\mathtt{u}(t-t_0)\}(s) = e^{-st_0}\mathcal{L}\{h(t)\}(s).$$

In this case $t_0 = 2$ and $h(t-2) = t$, then $h(t) = t+2$, and then

$$(s-1)(s+1)X(s) = 2s + e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right),$$

it is to say

$$(*)\quad X(s) = \frac{2s}{(s-1)(s+1)} + e^{-2s}\frac{2s+1}{(s-1)(s+1)s^2} = F(s) + G(s)e^{-2s}.$$

As we have

$$F(s) = \frac{2s}{(s-1)(s+1)} = \frac{1}{s-1} - \frac{1}{s+1} = \mathcal{L}\left\{e^t - e^{-t}\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s).$$

and

$$G(s) = \frac{2s+1}{(s-1)(s+1)s^2} = -\frac{1}{s^2} - \frac{2}{s} + \frac{3}{2(s-1)} + \frac{1}{2(s+1)}$$
$$= \mathcal{L}\left\{-t-2+3/2e^{t}+1/2e^{-t}\right\}(s) = \mathcal{L}\{g(t)\}(s).$$

We apply the last property to get

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\left\{f(t)\right\}(s) + e^{-2s}\mathcal{L}\{g(t)\}(s),$$

It is to say,

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\left\{e^{t} - e^{-t} + g(t-2)\mathtt{u}(t-2)\right\}(s).$$

Then, the solution is $\boxed{x(t) = e^{t} - e^{-t} + \frac{1}{2}\left(-2t+3e^{t-2}+e^{-t+2}\right)\mathtt{u}(t-2)}$.

**3. (3'5 points)** Find the Fourier series in its trigonometric form of the following periodic signal:

$$x(t) = t\,\mathtt{u}(t), \quad -\pi < t < \pi.$$

**Solution:** We have

$$\mathcal{F}\{x(t)\} = \frac{a_0}{2} + \sum_{k\geq 1} a_k \cos(kw_0 t) + \sum_{k\geq 1} b_k \sin(kw_0 t),$$

with $T = 2\pi$ and $w_0 = \frac{2\pi}{T} = 1$. We have

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{\pi}\int_0^{\pi} tdt\frac{\pi}{2},$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(kt)dt = \frac{1}{\pi}\int_0^{\pi} t\cos(kt)dt = \frac{(-1)^k-1}{\pi k^2},$$
$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(kt)dt = \frac{1}{\pi}\int_0^{\pi} t\sin(kt)dt = \frac{(-1)^{k+1}}{k},$$

where the last two integrals are solved by integration by parts.