LINEAR ALGEBRA

Extraordinary call

July 5th, 2019

1. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by the linear mapping defined by

$$f(x,y,z) = (x+y-z,\ y,\ 2x-z).$$

   a) (1 point) Obtain the coordinate matrix of f with respect to the basis $B = \{(-1,0,1),(0,1,0),(0,0,1)\}$.

   b) (1 point) Compute $f(1,2,-2)$by using the coordinate matrix you have obtained previously.

Solution: By using the definition of the linear mapping we have

a)

$$M_{B,B}(f) = M_{B,B_c} M_{B_c,B_c}(f) M_{B_c,B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

computing this we get

$$M_{B,B}(f) = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ -5 & 1 & -2 \end{pmatrix}.$$

b) In this case we have $(1,2,-2) = (-1,2,-1)_B$ therefore

$$f(1,2,-2) = M_{B,B}(f) \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 9 \end{pmatrix} \to (-5,2,9)_B = (5,2,4).$$

2. Consider the matrix:

$$A = \begin{pmatrix} b & -1 & b \\ 0 & 3 & 4 \\ 0 & -5 & -6 \end{pmatrix},$$

that depends on the parameter $b \in \mathbb{R}$.

   a) (1,25 points) Determine the values of b for whose the matrix A is diagonalizable over $\mathbb{R}$.

   b) (0,75 points) For the $b = 1$ case, find the matrices D (diagonal) and P (invertible) such that $A = PDP^{-1}$.

Solution: The characteristic polynomial of A is

a)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - b & 1 & -b \\ 0 & \lambda - 3 & -4 \\ 0 & 5 & \lambda + 6 \end{vmatrix} = (\lambda - b)(\lambda + 1)(\lambda + 2),$$

therefore the spectra of A is $\sigma(A) = \{b, -1, -2\}$. With this we can say:

- If $b \neq -1, -2$ then A is diagonalizable over $\mathbb{R}$.
- If $b = -1$ then $m_a(\lambda = -1) = 2$, so we need to compute $m_g(\lambda = -1)$. In fact

$$A - (-1)I = A + I = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 4 & 4 \\ 0 & -5 & -5 \end{pmatrix} \quad \to \quad m_g(\lambda = -1) = 2,$$

  hence A is diagonalizable over $\mathbb{R}$.

- If $b=-2$ then $m_a(\lambda=-2)=2$, so we need to compute $m_g(\lambda=-2)$. In this case we have

$$A-(-2)I = A+2I = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 5 & 4 \\ 0 & -5 & -4 \end{pmatrix} \quad \rightarrow \quad m_g(\lambda=-2)=1,$$

  hence A is not diagonalizable over $\mathbb{R}$.

b) If we set $b=1$ then $\sigma(A)=\{-2,-1,1\}$ and the eigenvectors are

$$v_{-2}=(3,4,-5), \quad v_{-1}=(1,1,-1), \quad \text{and} \quad v_1=(1,0,0).$$

With all these calculations we get

$$D=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P=\left(v_{-2}^t \mid v_{-1}^t \mid v_1^t \right) = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 1 & 0 \\ -5 & -1 & 0 \end{pmatrix}.$$

3. Let us consider the following initial value problem:

$$X'(t)=AX(t)+G(t), \quad X(0)=\begin{pmatrix} -2 \\ -2 \end{pmatrix},$$

where

$$A=\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}, \qquad G(t)=\begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}.$$

a) (0,75 points) Compute the matrix e^{At}.

b) (1,25 points) Solve the initial value problem by using the matrix e^{At} obtained previously.

Solution: Let us apply the basics from the theory:

a) The spectrum of A is $\sigma(A)=\{3,2\}$ and in fact $A=PDP^{-1}$ where

$$D=\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad P=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}.$$

Hence

$$e^{At}=Pe^{Dt}P^{-1}=\begin{pmatrix} -2e^{2t}+3e^{3t} & 3e^{2t}-3e^{3t} \\ -2e^{2t}+2e^{3t} & 3e^{2t}-2e^{3t} \end{pmatrix}.$$

b) Since the solution is of the form

$$X(t)=e^{At}X(0)+e^{At}\int_0^t e^{-As}G(s)\,ds$$

we obtain, after some basic calculations, that

$$e^{At}X(0)=\begin{pmatrix} -2e^{2t} \\ -2e^{2t} \end{pmatrix},$$

and

$$e^{At}\int_0^t e^{-As}G(s)\,ds=\begin{pmatrix} 2e^{2t}+(-2+3t)e^{3t} \\ 2e^{2t}+(-2+2t)e^{3t} \end{pmatrix}.$$

Hence the solution is

$$X(t)=\begin{pmatrix} (-2+3t)e^{3t} \\ (-2+2t)e^{3t} \end{pmatrix}.$$

4. In $\mathbb{R}^4$, under the standard scalar product, we consider the subspace U which has as a basis

$$\{(1,-1,0,0),\ (1,1,1,-2),\ (2,2,2,3)\}.$$

a) (1 point) Determine a basis for the orthogonal complement $U^\perp$ of U.

b) (1 point) Determine the orthogonal projection of the vector $b = (4, 2, 0, 1)$ onto U.

c) (1 point) Calculate the distance from the vector b to the subspace U.

Solution: Let us apply the basics from the theory:

a) The elemets of $U^\perp$ fulfill the equations:

$$U^\perp = \{(x, y, z, t) \in \mathbb{R}^4 : x - y = 0, x + y + z - 2t = 0, 2x + 2y + 2z + 3t = 0\},$$

then $U^\perp = \text{Span}\big((1, 1, -2, 0)\big)$.

b) Since $U^\perp$ has one element, then it is easy to get

$$proj_{U^\perp}(b) = A(A^TA)^{-1}A^Tb = (1, 1, -2, 0)^T,$$

where $A = (1, 1, -2, 0)^T$, therefore

$$proj_U(b) = b - proj_{U^\perp}(b) = (3, 1, 2, 1).$$

c) With this we have

$$d(b, U) = \|proj_{U^T}(b)\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}.$$

5. (1 point) Given the boolean function

$$f(w, x, y, z) = (x + z)\overline{y} + w\overline{x}.$$

Compute the *table of values* of f, and obtain the disjunctive and conjunctive normal forms of f.

Solution: The *table of values* of f is:

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
y	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
z	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
w	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$x + z$	0	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1
$\overline{y}$	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
$(x + z)\overline{y}$	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0
$w\overline{x}$	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
$f(x, y, z, w)$	0	**1**	**1**	**1**	0	**1**	0	**1**	**1**	**1**	**1**	**1**	0	0	0	0

With this we get that the disjunctive normal form of f is

$$f(w, x, y, z) = w\overline{xyz} + \overline{wxy}z + w\overline{xy}z + w\overline{x}y\overline{z} + w\overline{x}yz + \overline{w}x\overline{yz} + +wx\overline{yz} + \overline{w}x\overline{y}z + wx\overline{y}z,$$

and the conjunctive normal form is

$$f(w, x, y, z) = (w{+}x{+}y{+}z)(w{+}x{+}\overline{y}{+}z)(w{+}\overline{x}{+}\overline{y}{+}z)(w{+}\overline{x}{+}\overline{y}{+}z)(\overline{w}{+}\overline{x}{+}\overline{y}{+}z)(w{+}\overline{x}{+}\overline{y}{+}\overline{z})(\overline{w}{+}\overline{x}{+}\overline{y}{+}\overline{z}).$$

Duration of the exam: 2 hours and 30 minutes.