Multiple Meixner polynomials of second type with non-classical parameters
A first study

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Outline

1. The basics
   - Discrete Multiple Orthogonal Polynomials
   - The Nearest Neighbor Recurrence Relation
2. The example ($r = 2$)
   - Multiple Meixner OP of second type
   - The 1st Factorization
   - The 2nd Factorization
   - The properties of orthogonality
3. The zeros

THE BASICS

Discrete Multiple Orthogonal Polynomials

- One needs $r \geq 2$ measures $(\mu_i)_{i=1}^r$.
- The polynomials are indexed by a multi-index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, with length $|\vec{n}| = \sum n_i$.
- A type II multiple orthogonal polynomial is defined as a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}|$ so that for $i = 1, \ldots, r$,

$$\sum_{x=0}^{\infty} p_{\vec{n}}(x)\, x^k\, \omega_i(x) = 0, \qquad k = 0, 1, \ldots, n_i - 1.$$

- The index $\vec{n}$ is said to be normal if $P_{\vec{n}}$ is unique (up to a multiplicative factor) and has exactly degree $|\vec{n}|$.
- System of measures (μ_i) for which all the multi-indices are normal are known as perfect systems.

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The Nearest Neighbor Recurrence Relation (NNRR)

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + \beta_{\vec{n},k} P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x).$$

- The NNRR connects MOP of second type $P_{\vec{n}}$ with the polynomial of degree one higher $P_{\vec{n}+\vec{e}_k}$ and all the neighbors of degree one lower $P_{\vec{n}-\vec{e}_j}$ for $j = 1, \ldots, r$.
- Computing the coefficients:

$$a_{\vec{n},j} = \frac{\displaystyle\sum_{x=0}^{\infty} x^{n_j} P_{\vec{n}}(x)\,\omega_j(x)}{\displaystyle\sum_{x=0}^{\infty} x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x)\,\omega_j(x)},$$

- For $r = 2$, we have

$$p_{n_1+1,n_2}(x) - p_{n_1,n_2+1}(x) = p_{n_1,n_2}(x).$$

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$$p_{n_1+1,n_2}(x) - p_{n_1,n_2+1}(x) = c_{n_1,n_2}p_{n_1,n_2}(x).$$

THE EXAMPLE

Multiple Meixner Polynomials of second type

- The expression

$$M_{n_1,n_2}^{\beta_1,\beta_2,c}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\frac{c^{n_1+n_2-k_1-k_2}}{(c-1)^{n_1+n_2}}(-x)_{k_1+k_2}(\beta_2+x-k_1)_{n_2-k_2}(\beta_1+x)_{n_1-k_1}$$

- The coefficients of the NNRR:

$$a_{n_1,n_2,1} = cn_1\frac{(\beta_1+n_1-1)(\beta_2-n_1-\beta_1)}{(1-c)^2(n_2+\beta_2-n_1-\beta_1)},$$

$$a_{n_1,n_2,2} = cn_2\frac{(\beta_2+n_2-1)(\beta_1-n_2-\beta_2)}{(1-c)^2(n_1+\beta_1-n_2-\beta_2)},$$

$$\beta_{n_1,n_2,k} = \frac{n_1+n_2}{1-c}+(n_k+\beta_k)\frac{c}{1-c}, \quad k=1,2.$$

- $c_{n_1,n_2} = \dfrac{c(n_2-n_1+\beta_2-\beta_1)}{1-c}.$

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- $c_{n_1,n_2} = \dfrac{c(n_2-n_1+\beta_2-\beta_1)}{1-c}.$

The 1st factorization

The coefficients of the NNRR:

$$a_{N,n_2,1} = cn_1 \frac{(\beta_1 + N - 1)(\beta_2 - N - \beta_1)}{(1-c)^2(n_2 + \beta_2 - N - \beta_1)},$$

$$a_{N,n_2,2} = cn_2 \frac{(\beta_2 + n_2 - 1)(\beta_1 - n_2 - \beta_2)}{(1-c)^2(N + \beta_1 - n_2 - \beta_2)},$$

$$c_{n_1,n_2} = \frac{c(n_2 - N + \beta_2 - \beta_1)}{1-c}$$

The case $\beta_1 = 1 - N$

$$M_{(N,n_2)}^{1-N,\beta_2,c}(x) = M_{(N,0)}^{1-N,\beta_2,c}(x) R_{(0,n_2)}(x) = (x-N+1)_N M_{n_2}^{\beta_2+N}(x-N)$$

The 2nd factorization

the 2nd case: Moving through the first index

$$M^{1-N,\beta_2,c}_{(N+n_1,0)}(x) = M^{1-N,\beta_2,c}_{(N,0)}(x)U_{(n_1,0)}(x) = (x-N+1)_N M^{N+1}_{n_1}(x-N)$$

The factorization and the orthogonality

$$p_{n_1+1,n_2}(x) - p_{n_1,n_2+1}(x) = (\beta_{n_1,n_2,2} - \beta_{n_1,n_2,1})p_{n_1,n_2}(x).$$

The general case

$$M^{1-N,\beta_2,c}_{(N+n_1,n_2)}(x) = (x-N+1)_N M^{1+N,\beta_2+N,c}_{(n_1,n_2)}(x-N)$$

The properties of orthogonality for the case $\beta_1 = 1-N$

$$\sum_{x=0}^{N} M^{1-N,\beta_2,c}_{(n_1,n_2)}(x)x^k \frac{(1-N)_x c^x}{\Gamma(x+1)} + \sum_{x=0}^{\infty}(\Delta^N M^{1-N,\beta_2,c}_{(n_1,n_2)}(x))(\Delta^N x^k)\, c^x = 0, \qquad 0 \le k \le n_1 - 1,$$

$$\sum_{x=0}^{\infty} M^{1-N,\beta_2,c}_{(n_1,n_2)}(x)x^k \frac{(\beta_2)_x c^x}{\Gamma(x+1)} = 0, \qquad 0 \le k \le n_2 - 1.$$

Sketch of the proof

Identity (by W. Van Assche)

$$\Delta M_{(m_1,m_2)}^{\beta_1,\beta_2,c}(x) = \frac{m_1(\beta_1-\beta_2-m_2)}{\beta_1-\beta_2} M_{(m_1-1,m_2)}^{\beta_1+1,\beta_2,c}(x) - \frac{m_2(\beta_2-\beta_1-m_1)}{\beta_1-\beta_2} M_{(m_1,m_2-1)}^{\beta_1,\beta_2+1,c}(x),$$

Consequence:

$$\Delta^N M_{(n_1,n_2)}^{1-N,\beta_2,c}(x) = \sum_{j=0}^{N} \mathfrak{a}_j M_{(n_1-j,n_{2,j})}^{1-N+j,\widetilde{\beta}_2,c}(x).$$

THE ZEROS

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