A connection between the Legendre polynomials and the Riemann Zeta function

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Work supported by MCeI grant MTM2009-12740-C03-01

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Sevilla, September 16-20 2013

Outline

THE BASICS

Results on Number Theory

- Let x be a real number. If there exist (p_n) and (q_n) so that

$$q_n x - p_n \neq 0 \quad \forall n \quad \& \quad \lim_{n\to\infty} q_n x - p_n = 0 \quad \Rightarrow \quad x \in \mathbb{I}.$$

- Irrationality measure

$$r(x) = \inf\{r \in \mathbb{R} : |x - \tfrac{a}{b}| < \tfrac{1}{b^r} \text{ fin. num. int. sol. in } (a,b)\}.$$

Observe that if $|x - p_n/q_n| = \mathcal{O}(1/q_n^{s+1})$ with $0 < s < 1$ and $q_n < q_{n+1} < q_n^{1+o(1)}$ then it verifies

$$1 + s < r(x) < 1 + 1/s.$$

- Hermite-Padé approximants: Type I, and Type II.
- **[REF]** R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$, in Journées arithmétiques (Luminy, 1978)*, Astérisque **61** (1979), 11–13.

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The Legendre polynomials

$$P_n(x) = \frac{1}{\Gamma(n+1)} \frac{d^n}{dx^n} \left(x^n(1-x)^n\right)$$

- The property of orthogonality on $[0, 1]$

$$\int_0^1 P_n(x) x^k \, dx = 0, \qquad 0 \le k < n.$$

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$$f_1(z) = \int_0^1 \frac{dx}{z-x}, \quad f_2(z) = -\int_0^1 \log x \frac{dx}{z-x}.$$

and the Hermite-Padé approximants problem is:

$$A_n(z) - B_n(z)\log(z) = O((1-z)^{n+1}), \quad z \to 1,$$

$$A_n(z) f_1(z) + B_n(z) f_2(z) - C_n(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

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$$\zeta(2), \zeta(3), \zeta(5), \ldots$$

The construction (I)

- The multiple integral $(0 < \ell \leq n)$

$$\mathcal{F}_{\ell,n}^{m} := \int_0^1 \cdots \int_0^1 \frac{(1-x_1)^m \cdots (1-x_\ell)^m P_m(x_{\ell+1}) \cdots P_m(x_n)}{1 - x_1 \cdots x_n}\, dx_1 \cdots dx_n$$

- After some calculations

$$d_m^n \mathcal{F}_{n-\ell,n}^{m} = A_{0,m,n}^{(\ell)} + \sum_{\xi=2}^{n} A_{\xi,m,n}^{(\ell)} \zeta(\xi)$$

where

$$d_m := L.C.M.(1, 2, \ldots, m).$$

- Some coefficients

$$A_{n,m,n}^{(\ell)} = d_m^n \,{}_{n+1}F_n \left(\begin{matrix} -m & \cdots & -m & m+1 & \cdots & m+1 \\ 1 & & \cdots & \cdots & & 1 \end{matrix} \middle| 1 \right),$$

with $n-\ell$ parameters equal to $-m$ and ℓ parameters equal to $m+1$.

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$n = 3$, $\ell = 1$

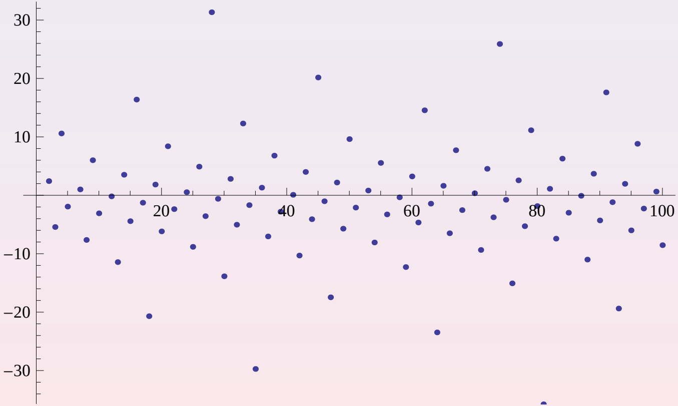

Figure: Ratio $A^{(1)}_{0,m,3}/A^{(1)}_{3,m,3}$, $1 \leq m \leq 100$

$n = 3,\ \ell = 2$

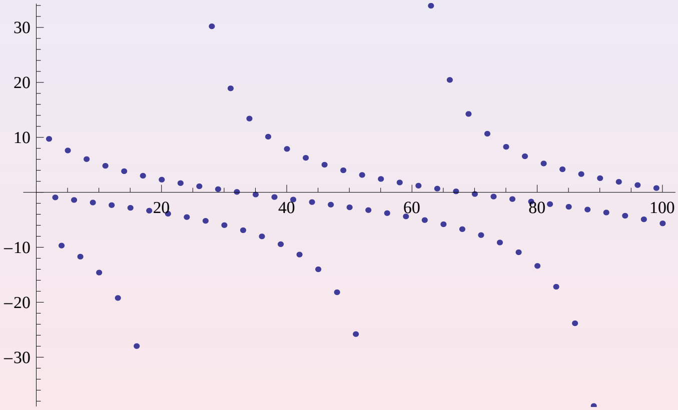

Figure: Ratio $A^{(2)}_{0,m,3}/A^{(2)}_{3,m,3}$, $1 \leq m \leq 100$

Interesting numerical facts

Behaviour of Ratios $A^{(\ell)}_{\xi,m,n}/A^{(\ell)}_{n,m,n}$

- If n is even and ξ even then the ratio tends to a constant.
- If n is even and ξ odd then the ratio tends to zero.
- If n is odd and ξ odd then the ratio tends to a constant.
- If n is odd ξ is even then the ratio oscillates.

The approximation of $\zeta(n) \equiv \zeta(n)_{m,M}$

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The construction (II)

$$\begin{cases} d_m^n \mathcal{F}_{n-1,n}^m = & A_{0,m,n}^{(1)} + \sum_{\xi=2}^n A_{\xi,m,n}^{(1)} \zeta(\xi) \\ d_m^n \mathcal{F}_{n-2,n}^m = & A_{0,m,n}^{(2)} + \sum_{\xi=2}^n A_{\xi,m,n}^{(2)} \zeta(\xi) \\ \vdots & \\ d_m^n \mathcal{F}_{1,n}^m = & A_{0,m,n}^{(n-1)} + \sum_{\xi=2}^n A_{\xi,m,n}^{(n-1)} \zeta(\xi) \end{cases}$$

OPEN PROBLEM

To find a Hermite-Padé approximation problem of a set of functions related to such rational approximation at some value.

CONJECTURE

The numbers $\zeta(2), \zeta(3), \zeta(4), \ldots$ are algebraically independent over $\mathbb{Q}$.

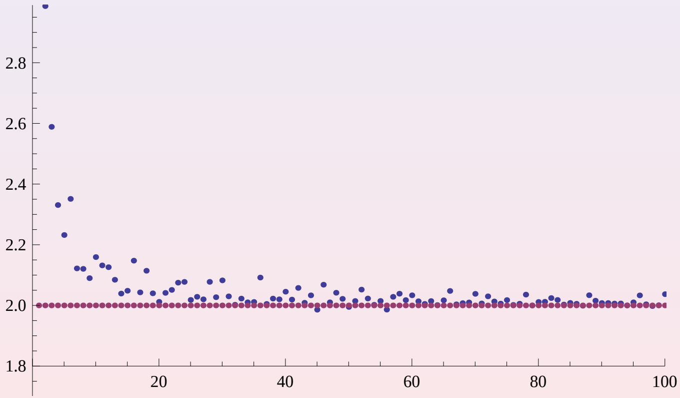
Numerical computation

$$\zeta(3)_{20,4} - \zeta(3) = \mathcal{O}(10^{-71})$$

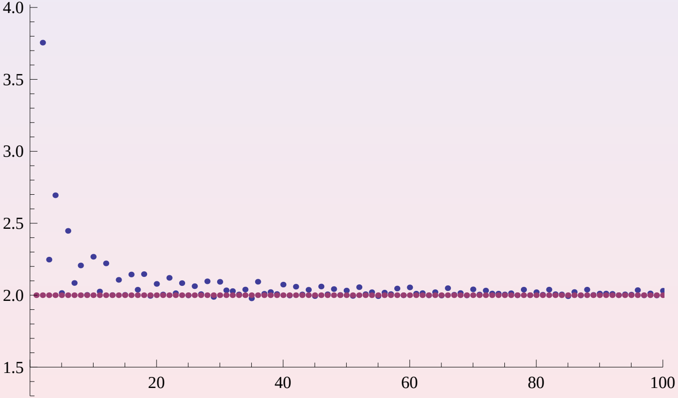
$$\zeta(3)_{30,4} - \zeta(3) = \mathcal{O}(10^{-106})$$

$$\zeta(3)_{25,6} - \zeta(3) = \mathcal{O}(10^{-112})$$

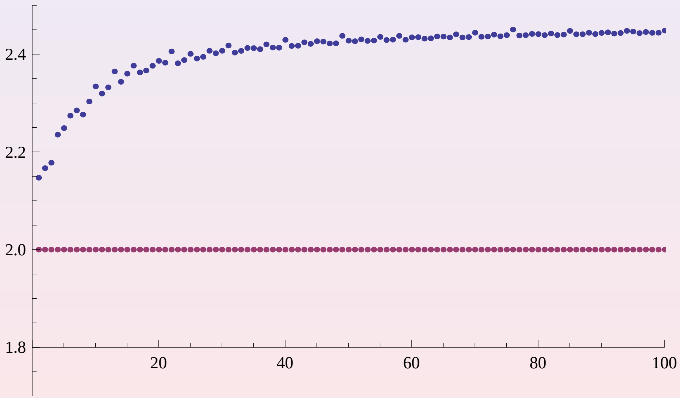
Irrationality measure of π^2



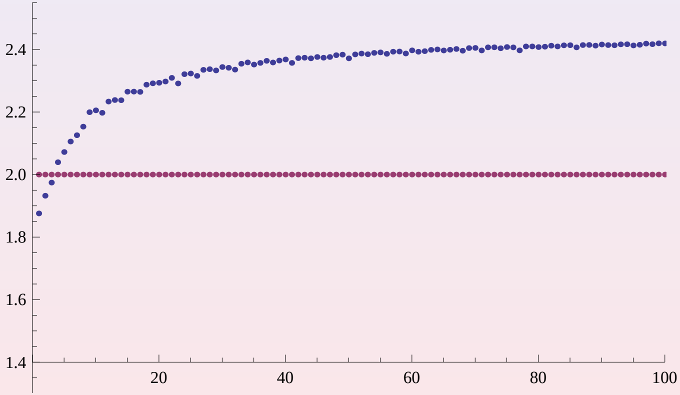
Irrationality measure of $\zeta(3)$



Irrationality measure of π^2 (diagonal mode)



Irrationality measure of $\zeta(3)$ (diagonal mode)



$$\gamma = \sum_{n\geq 2} \frac{(-1)^n \zeta(n)}{n}.$$

Then for any N, we consider the better rational approximants of $\zeta(2), \ldots, \zeta(N)$ and apply such sum getting a worth approximation.

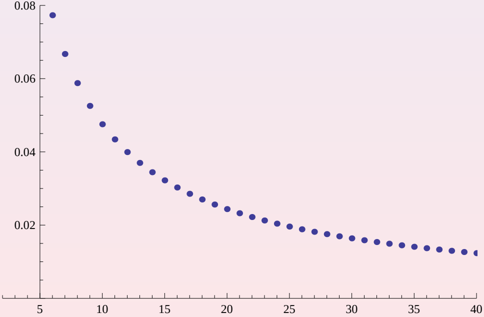


Figure: $\left|\gamma - \sum_{n=2}^{N} \frac{(-1)^n \zeta(n)_{10,N}}{n}\right|$

$$\zeta_q(2), \zeta_q(3), \zeta_q(5), \ldots$$

The key identity

Remark: From this identity ...

$$\mathcal{F}^m_{\ell,n} = (-1)^{m(n-\ell)}(\Gamma(m+1))^\ell \sum_{s=0}^{\infty} \left(\frac{\Gamma(s+1)}{\Gamma(s+m+2)}\right)^n \left(\frac{\Gamma(s+1)}{\Gamma(s-m+1)}\right)^{n-\ell}$$

... we consider the q-analog

$$\frac{\prod_{k=0}^{m-1}\left(\frac{q^{-k+s}-1}{q-1}\right)^{n-\ell}}{\prod_{k=0}^{m+1}\left(\frac{q^{k+s}-1}{q-1}\right)^{n}}$$

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THANK YOU
FOR YOUR ATTENTION !!