Factorization of the hypergeometric-type difference equation on the uniform lattice

R. Costas Santos

Work supported by DGES grant BFM 2003-06335-C03

Universidad Carlos III de Madrid

Collaboration with: R. Alvarez-Nodarse   N.M. Atakishiyev

Universidad de Sevilla   UNAM

0. Motivation

1 The study of discrete system has attracted the attention of many authors in the last years. Of special interest are the discrete analogs of the quantum harmonic oscillators.

0. Motivation

1. The study of discrete system has attracted the attention of many authors in the last years. Of special interest are the discrete analogs of the quantum harmonic oscillators.

2. This classical FM is based on the existence of the so-called raising/lowering operators, which allow to obtain the explicit solutions.

$$a^{\downarrow}(s)\Phi_n(s) = u_n\Phi_{n+1}(s),\ a^{\uparrow}(s)\Phi_n(s) = d_n\Phi_{n-1}(s).$$

0. Motivation

1 The study of discrete system has attracted the attention of many authors in the last years. Of special interest are the discrete analogs of the quantum harmonic oscillators.

2 This classical FM is based on the existence of the so-called raising/lowering operators, which allow to obtain the explicit solutions.

$$a^{\downarrow}(s)\Phi_n(s) = u_n\Phi_{n+1}(s),\ a^{\uparrow}(s)\Phi_n(s) = d_n\Phi_{n-1}(s).$$

3 Our main purpose is use the FM with difference equations on the uniform lattice $x(s) = s$ in an unified form. This paper provides a background to analyze the q-linear case (since limit $q \uparrow 1$).

1. Preliminaries

1 We start with a **DE of hypergeometric-type**

$$\sigma(s)\Delta\nabla y(s) + \tau(s)\Delta y(s) + \lambda y(s) = 0.$$

1. Preliminaries

1. We start with a **DE of hypergeometric-type**

$$\sigma(s)\Delta\nabla y(s) + \tau(s)\Delta y(s) + \lambda y(s) = 0.$$

2. Let's $\{P_n\}_{n\in\mathbb{N}_0}$ be the SPO with respect $\rho(s)$, i.e.

$$\langle P_n, P_m\rangle = \sum_{s=a}^{b} P_n(s)P_m^*(s)\rho(s) = d_n^2\delta_{n,m}.$$

1. Preliminaries

$\boxed{1}$ We start with a **DE of hypergeometric-type**

$$\sigma(s)\Delta\nabla y(s) + \tau(s)\Delta y(s) + \lambda y(s) = 0.$$

$\boxed{2}$ Let's $\{P_n\}_{n\in\mathbb{N}_0}$ be the SPO with respect $\rho(s)$, i.e.

$$\langle P_n, P_m\rangle = \sum_{s=a}^{b} P_n(s)P_m^*(s)\rho(s) = d_n^2\delta_{n,m}.$$

$\boxed{3}$ Orthonormal funct.: $\Phi_n(s) = d_n^{-1}\sqrt{\rho(s)}P_n(s)$.

1. Preliminaries

1 We start with a **DE of hypergeometric-type**

$$\sigma(s)\Delta\nabla y(s) + \tau(s)\Delta y(s) + \lambda y(s) = 0.$$

2 Let's $\{P_n\}_{n\in\mathbb{N}_0}$ be the SPO with respect $\rho(s)$, i.e.

$$\langle P_n, P_m\rangle = \sum_{s=a}^{b} P_n(s)P_m^*(s)\rho(s) = d_n^2\delta_{n,m}.$$

3 Orthonormal funct.: $\Phi_n(s) = d_n^{-1}\sqrt{\rho(s)}P_n(s)$.

4 The **hamiltonian** associated to this functions

$$\mathfrak{h}_1(s) = -\nu(s-1)\, e^{-\partial_s} - \nu(s)\, e^{\partial_s} + [2\sigma(s) + \tau(s)]I_d,$$

1. Preliminaries

$\boxed{5}$ $\mathfrak{h}_2(s) = C_a^2 \mathfrak{h}_1(s) + E, \quad C_a, E\; const,$

where

$$\nu(s) = \sqrt{\sigma(s+1)[\sigma(s)+\tau(s)]}$$

$$\begin{aligned}
\Delta f(s) &= f(s+1) - f(s),\ i.e.\ \ \Delta = e^{\partial_s} - I_d,\\
\nabla f(s) &= f(s) - f(s-1),\ i.e.\ \ \nabla = I_d - e^{-\partial_s},\\
e^{\alpha\partial_s} f(s) &= f(s+\alpha), \qquad\qquad I_d = \text{identity}.
\end{aligned}$$

1. Preliminaries

5 $\mathfrak{h}_2(s) = C_a^2 \mathfrak{h}_1(s) + E, \quad C_a, E\ const,$

where

$$\nu(s) = \sqrt{\sigma(s+1)[\sigma(s)+\tau(s)]}$$

$$\Delta f(s) = f(s+1) - f(s),\ i.e.\ \ \Delta = e^{\partial_s} - I_d,$$
$$\nabla f(s) = f(s) - f(s-1),\ i.e.\ \ \nabla = I_d - e^{-\partial_s},$$
$$e^{\alpha\partial_s} f(s) = f(s+\alpha), \qquad I_d = \text{identity}.$$

6 Let's consider the following set

$$\mathcal{A} := \{\mathcal{D} : \langle \mathcal{D} f, g\rangle = \langle f, \mathcal{D} g\rangle \quad \forall\, f, g\}.$$

1. Preliminaries

5 $\mathfrak{h}_2(s) = C_a^2 \mathfrak{h}_1(s) + E, \quad C_a, E\; const,$

where

$$\nu(s) = \sqrt{\sigma(s+1)[\sigma(s)+\tau(s)]}$$

$$\begin{aligned} \Delta f(s) &= f(s+1) - f(s), \; i.e. \;\; \Delta = e^{\partial_s} - I_d, \\ \nabla f(s) &= f(s) - f(s-1), \; i.e. \;\; \nabla = I_d - e^{-\partial_s}, \\ e^{\alpha \partial_s} f(s) &= f(s+\alpha), \qquad\qquad I_d = \text{identity}. \end{aligned}$$

6 Let's consider the following set

$$\mathcal{A} := \{\mathcal{D} : \langle \mathcal{D} f, g \rangle = \langle f, \mathcal{D} g \rangle \quad \forall\, f, g\}.$$

◇ $\mathfrak{h}_1(s),\ I_d \in \mathcal{A}$, hence $\mathfrak{h}_2(s) \in \mathcal{A}$.

3. The Problem

- Find two operators $\mathfrak{a}(s)$, $\mathfrak{b}(s)$ such that

$$\mathfrak{b}(s)\mathfrak{a}(s) = \mathfrak{h}_2(s) \quad \text{and} \quad [\mathfrak{b}(s), \mathfrak{a}(s)] = I_d.$$

3. The Problem

- Find two operators $\mathfrak{a}(s)$, $\mathfrak{b}(s)$ such that

$$\mathfrak{b}(s)\mathfrak{a}(s) = \mathfrak{h}_2(s) \quad \text{and} \quad [\mathfrak{b}(s), \mathfrak{a}(s)] = I_d.$$

Def: For every $\alpha \in \mathbb{R}$, let's define the operators

$$\mathfrak{a}_\alpha^\downarrow(s) := e^{-\alpha\partial_s}\left(e^{\partial_s}\sqrt{\sigma(s)} - \sqrt{\sigma(s)+\tau(s)}\, I_d\right),$$

$$\mathfrak{a}_\alpha^\uparrow(s) := \left(\sqrt{\sigma(s)}e^{-\partial_s} - \sqrt{\sigma(s)+\tau(s)}\, I_d\right)e^{\alpha\partial_s},$$

3. The Problem

- Find two operators $\mathfrak{a}(s)$, $\mathfrak{b}(s)$ such that

$$\mathfrak{b}(s)\mathfrak{a}(s) = \mathfrak{h}_2(s) \quad \text{and} \quad [\mathfrak{b}(s), \mathfrak{a}(s)] = I_d.$$

Def: For every $\alpha \in \mathbb{R}$, let's define the operators

$$\mathfrak{a}^{\downarrow}_{\alpha}(s) := e^{-\alpha\partial_s}\left(e^{\partial_s}\sqrt{\sigma(s)} - \sqrt{\sigma(s)+\tau(s)}\, I_d\right),$$

$$\mathfrak{a}^{\uparrow}_{\alpha}(s) := \left(\sqrt{\sigma(s)}e^{-\partial_s} - \sqrt{\sigma(s)+\tau(s)}\, I_d\right)e^{\alpha\partial_s},$$

Lemma:

$$\mathfrak{a}^{\uparrow}_{\alpha}(s)\mathfrak{a}^{\downarrow}_{\alpha}(s) = \mathfrak{h}_1(s), \quad \forall\, \alpha \in \mathbb{R}.$$

4. A characterization theorem

Theorem:(Costas-Santos et al.)
The operators $b(s) = \mathfrak{a}_\alpha^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_\alpha^{\downarrow}(s)$, factorize the Hamiltonian $\mathfrak{h}_1(s)$ and satisfy the commutation relation $[a(s), b(s)] = \Lambda$ for a certain complex number Λ, if and only if the following two conditions holds:

1. $$\frac{\sigma(s-\alpha)[\sigma(s-\alpha)+\tau(s-\alpha)]}{\sigma(s)[\sigma(s-1)+\tau(s-1)]} = 1,$$

2. $\sigma(s-\alpha+1) + \sigma(s-\alpha) + \tau(s-\alpha) - 2\sigma(s) - \tau(s) = \Lambda.$

4. A characterization theorem

Theorem:(Costas-Santos et al.)
The operators $b(s) = \mathfrak{a}_{\alpha}^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_{\alpha}^{\downarrow}(s)$, factorize the Hamiltonian $\mathfrak{h}_1(s)$ and satisfy the commutation relation $[a(s), b(s)] = \Lambda$ for a certain complex number Λ, if and only if the following two conditions holds:

1. $$\frac{\sigma(s-\alpha)[\sigma(s-\alpha)+\tau(s-\alpha)]}{\sigma(s)[\sigma(s-1)+\tau(s-1)]} = 1,$$

2. $\sigma(s-\alpha+1)+\sigma(s-\alpha)+\tau(s-\alpha)-2\sigma(s)-\tau(s) = \Lambda.$

$\diamond$ If $\alpha = 0 \;\Rightarrow\; \sigma(s)+\tau(s) = const.$ **Case Charlier**

4. A characterization theorem

Theorem:(Costas-Santos et al.)
The operators $b(s) = \mathfrak{a}_{\alpha}^{\uparrow}(s)$ and $a(s) = \mathfrak{a}_{\alpha}^{\downarrow}(s)$, factorize the Hamiltonian $\mathfrak{h}_1(s)$ and satisfy the commutation relation $[a(s), b(s)] = \Lambda$ for a certain complex number Λ, if and only if the following two conditions holds:

1. $$\frac{\sigma(s-\alpha)[\sigma(s-\alpha)+\tau(s-\alpha)]}{\sigma(s)[\sigma(s-1)+\tau(s-1)]} = 1,$$

2. $\sigma(s-\alpha+1)+\sigma(s-\alpha)+\tau(s-\alpha)-2\sigma(s)-\tau(s) = \Lambda.$

$\diamond$ If $\alpha = 0 \;\Rightarrow\; \sigma(s)+\tau(s) = const.$ **Case Charlier**

$\diamond$ If $\alpha = 1 \;\Rightarrow\; \sigma(s) = const.$

5. The Case Charlier ($\alpha = 0$)

The **Hamiltonian**

$$\mathfrak{h}_1^C(s) = -\sqrt{s\mu}\, e^{-\partial_s} - \sqrt{(s+1)\mu}\, e^{\partial_s} + (s+\mu)I_d.$$

Orthonormal functions: $\Phi_n^C(s) = \sqrt{\frac{e^{-\mu}\mu^{s-n}}{s!\,n!}}\, C_n^\mu(s),\ \mu > 0.$

The Operators ($\Lambda = 1$)

$$\mathfrak{a}(s) = \sqrt{s+1}\, e^{\partial_s} - \sqrt{\mu}\, I_d,\ \mathfrak{b}(s) = \sqrt{s} e^{-\partial_s} - \sqrt{\mu}\, I_d,$$

$$\mathfrak{a}(s)\Phi_n^C(s) = \sqrt{n}\Phi_{n-1}^C(s), \quad \mathfrak{b}(s)\Phi_n^C(s) = \sqrt{n+1}\Phi_{n+1}^C(s),$$

$$\Phi_n^C(s) = \frac{1}{\sqrt{n!}}\left[\sqrt{s}\, e^{\partial_s} - \sqrt{\mu}\, I\right]^n \left(\sqrt{\frac{e^{-\mu}\mu^s}{s!}}\right).$$

This example constitutes a discrete analog of the quantum harmonic oscillator.

6. Other special cases

Now, we know that Charlier functions have a closed simple oscillator algebra. But ...

What to do in the other classical cases?.

6. Other special cases

Now, we know that Charlier functions have a closed simple oscillator algebra. But ...

What to do in the other classical cases?.

- ▷ Let's consider the following operators

$$a(s) = \sqrt{\sigma(s+1)}\, e^{\frac{1}{2}\partial_s} - \sqrt{\sigma(s-1)+\tau(s-1)}\, e^{-\frac{1}{2}\partial_s},$$

$$a^{\dagger}(s) = e^{-\frac{1}{2}\partial_s}\sqrt{\sigma(s+1)} - e^{\frac{1}{2}\partial_s}\sqrt{\sigma(s-1)+\tau(s-1)}.$$

6. Other special cases

Now, we know that Charlier functions have a closed simple oscillator algebra. But ...

What to do in the other classical cases?.

- Let's consider the following operators

$$a(s) = \sqrt{\sigma(s+1)}\, e^{\frac{1}{2}\partial_s} - \sqrt{\sigma(s-1)+\tau(s-1)}\, e^{-\frac{1}{2}\partial_s},$$

$$a^{\dagger}(s) = e^{-\frac{1}{2}\partial_s}\sqrt{\sigma(s+1)} - e^{\frac{1}{2}\partial_s}\sqrt{\sigma(s-1)+\tau(s-1)}.$$

- $\mathfrak{h}_1(s) = a(s)a^{+}(s) + (\tau' - \sigma'')I_d.$

6. Other special cases

Now, we know that Charlier functions have a closed simple oscillator algebra. But ...

What to do in the other classical cases?.

- ▷ Let's consider the following operators

$$a(s) = \sqrt{\sigma(s+1)}\, e^{\frac{1}{2}\partial_s} - \sqrt{\sigma(s-1)+\tau(s-1)}\, e^{-\frac{1}{2}\partial_s},$$

$$a^{\dagger}(s) = e^{-\frac{1}{2}\partial_s}\sqrt{\sigma(s+1)} - e^{\frac{1}{2}\partial_s}\sqrt{\sigma(s-1)+\tau(s-1)}.$$

- ▷ $\mathfrak{h}_1(s) = a(s)a^{+}(s) + (\tau' - \sigma'')I_d.$

- ▶ $\mathfrak{c}(s) = C_a C_b \mathfrak{a}(s) e^{-\frac{1}{2}\partial_s}\sqrt{\sigma(s+1)},$

  $\mathfrak{c}^{\dagger}(s) = \sqrt{\sigma(s+1)} e^{\frac{1}{2}\partial_s}\mathfrak{a}^{\dagger}(s) C_b C_a.$

7. Observations

Theorem: (Costas-Santos et al.)
The operators $\mathfrak{c}(s)$ and $\mathfrak{c}^{\dagger}(s)$ are mutually adjoint.

7. Observations

Theorem: (Costas-Santos et al.)
The operators $\mathfrak{c}(s)$ and $\mathfrak{c}^\dagger(s)$ are mutually adjoint.

- If $\sigma'' = 0$ the **spectrum is linear**, i.e. $\lambda_n = -n\tau'$.

7. Observations

Theorem: (**Costas-Santos et al.**)
The operators $\mathfrak{c}(s)$ and $\mathfrak{c}^{\dagger}(s)$ are mutually adjoint.

- ▷ If $\sigma'' = 0$ the **spectrum is linear**, i.e. $\lambda_n = -n\tau'$.
- ▷ Moreover, if $\sigma'' = 0$ then $\mathfrak{h}_2(s)$, $\mathfrak{c}(s)$, $\mathfrak{c}^{\dagger}(s)$ and I_d form a closed algebra, i.e.

$$\begin{aligned} [\mathfrak{h}_2(s), \mathfrak{c}(s)] &= -\mathfrak{c}(s) + C_1\mathfrak{h}_2(s) + C_2 I_d, \\ [\mathfrak{h}_2(s), \mathfrak{c}^{\dagger}(s)] &= \mathfrak{c}^{\dagger}(s) - C_1\mathfrak{h}_2(s) - C_2 I_d, \\ [\mathfrak{c}(s), \mathfrak{c}^{\dagger}(s)] &= C_3\mathfrak{h}_2(s) + C_4 I_d. \end{aligned}$$

7. Observations

Theorem: (Costas-Santos et al.)
The operators $\mathfrak{c}(s)$ and $\mathfrak{c}^\dagger(s)$ are mutually adjoint.

- ▷ If $\sigma'' = 0$ the **spectrum is linear**, i.e. $\lambda_n = -n\tau'$.
- ▷ Moreover, if $\sigma'' = 0$ then $\mathfrak{h}_2(s)$, $\mathfrak{c}(s)$, $\mathfrak{c}^\dagger(s)$ and I_d form a closed algebra, i.e.

$$\begin{aligned}[\mathfrak{h}_2(s), \mathfrak{c}(s)] &= -\mathfrak{c}(s) + C_1 \mathfrak{h}_2(s) + C_2 I_d, \\ [\mathfrak{h}_2(s), \mathfrak{c}^\dagger(s)] &= \mathfrak{c}^\dagger(s) - C_1 \mathfrak{h}_2(s) - C_2 I_d, \\ [\mathfrak{c}(s), \mathfrak{c}^\dagger(s)] &= C_3 \mathfrak{h}_2(s) + C_4 I_d.\end{aligned}$$

- ▶ The cases **Meixner**, **Kravchuk** and **Charlier** satisfy $\sigma'' = 0$.

8. Construction of the Algebra

Let's consider the following operators

$$
\begin{aligned}
K_0(s) &= \mathfrak{h}_2(s)(-\tau' C_a^2)^{-1} \\
K_-(s) &= -\tau' C_a^2\, c(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right), \\
K_+(s) &= -\tau' C_a^2\, c^+(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right).
\end{aligned}
$$

8. Construction of the Algebra

Let's consider the following operators

$$\begin{aligned} K_0(s) &= \mathfrak{h}_2(s)(-\tau' C_a^2)^{-1} \\ K_-(s) &= -\tau' C_a^2\, c(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right), \\ K_+(s) &= -\tau' C_a^2\, c^+(s) - C_b C_a \sigma'(0)\left(\mathfrak{h}_2(s) - \tau' C_a^2 - E\right). \end{aligned}$$

Then

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \;\text{ y }\; [K_-(s), K_+(s)] = A_0 K_0(s) + A_1,$$

where $C_a^2 \tau' = -1$,

$$\begin{aligned} A_0 &= 2\sigma'(0) C_b^2 C_a^2 (\sigma'(0) + \tau') \\ A_1 &= -E A_0 + C_b^2 C_a^2 [\sigma'(0)\tau(0) - \sigma(0)\tau']. \end{aligned}$$

9. Case $A_0 > 0$

We choose C_b, E in such a way that $A_0 = 2$, $A_1 = 1$.

$$C_b^2 = \frac{-\tau'}{\sigma'(0)[\tau' + \sigma'(0)]}, \quad E = -\frac{C_b^2[\sigma'(0)\tau(0) - \sigma(0)\tau']}{2\tau'},$$

obtaining

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \text{ and } [K_-(s), K_+(s)] = 2K_0(s),$$

which corresponds with the **Lie algebra** $\mathrm{Sp}(2, \mathfrak{R})$.

9. Case $A_0 > 0$

We choose C_b, E in such a way that $A_0 = 2$, $A_1 = 1$.

$$C_b^2 = \frac{-\tau'}{\sigma'(0)[\tau' + \sigma'(0)]}, \quad E = -\frac{C_b^2[\sigma'(0)\tau(0) - \sigma(0)\tau']}{2\tau'},$$

obtaining

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \text{ and } [K_-(s), K_+(s)] = 2K_0(s),$$

which corresponds with the **Lie algebra** Sp$(2, \mathfrak{R})$.

◇ The **Casimir operator**

$$K^2(s) = K_0^2(s) - K_0(s) - K_+(s)K_-(s) = E(E-1)I_d.$$

9. Case $A_0 > 0$

We choose C_b, E in such a way that $A_0 = 2$, $A_1 = 1$.

$$C_b^2 = \frac{-\tau'}{\sigma'(0)[\tau' + \sigma'(0)]}, \quad E = -\frac{C_b^2[\sigma'(0)\tau(0) - \sigma(0)\tau']}{2\tau'},$$

obtaining

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \text{ and } [K_-(s), K_+(s)] = 2K_0(s),$$

which corresponds with the **Lie algebra** $\mathrm{Sp}(2, \mathfrak{R})$.

- ◇ The **Casimir operator**

$$K^2(s) = K_0^2(s) - K_0(s) - K_+(s)K_-(s) = E(E-1)I_d.$$

- ◇ Example: Case Meixner.

10. Case Meixner

The **Meixner functions**

$$\Phi_n^M(s) = \mu^{(s-n)/2}(1-\mu)^{\gamma/2+n}\sqrt{\frac{(\gamma)_s}{s!n!(\gamma)_n}}\,M_n^{\gamma,\mu}(s),\; n \geq 0,$$

and **the hamiltonian**

$$\mathfrak{h}_1^M(s) = -\sqrt{\mu s(s+\gamma-1)}e^{-\partial_s} - \sqrt{\mu(s+1)(s+\gamma)}e^{\partial_s} + (s+\mu\,(s+\gamma))\,I_d,$$

Then the operators $K_+(s)$ and $K_-(s)$ satisfy

$$K_+(s)\Phi_n^M(s) = \sqrt{(n+1)(n+\gamma)}\Phi_{n+1}^M(s),\; K_-(s)\Phi_n^M(s) = \sqrt{n(n+\gamma-1)}\Phi_{n-1}^M(s).$$

$$\Phi_n^M(s) = \sqrt{\frac{(1-\mu)^\gamma}{n!\Gamma(\gamma+n)}}\,(K_+(s))^n\left[\sqrt{\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(s+1)}}\right].$$

¡¡ K_+ is a 2nd order difference operator !!

11. Case Kravchuk ($A_0 < 0$)

◇ We can choose C_b and E such that

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \text{ and } [K_+(s), K_-(s)] = 2K_0(s),$$

which corresponds with **Lie algebra** so(3).

11. Case Kravchuk ($A_0 < 0$)

- We can choose C_b and E such that

$$[K_0(s), K_\pm(s)] = \pm K_\pm(s) \text{ and } [K_+(s), K_-(s)] = 2K_0(s),$$

which corresponds with **Lie algebra** so(3).

The hamiltonian

$$\mathfrak{h}_1^K(s) = -\frac{\sqrt{ps(N-s+1)}}{\sqrt{1-p}} e^{-\partial_s} + \frac{Np+s-2ps}{1-p} I_d - \frac{\sqrt{p(s+1)(N-s)}}{\sqrt{1-p}} e^{\partial_s},$$

$$\Phi_n^K(s) = \sqrt{\frac{(N-n)!(1-p)^{N-n}}{N!p^n}} (K_+(s))^n \left[\binom{N}{s}\left(\frac{p}{1-p}\right)^s\right].$$

where K_+ is, again, a 2nd order difference operator.

Conclusions and remarks

- This method allow us to understand different cases, which were studied one to one, in a simple way.

Conclusions and remarks

- This method allow us to understand different cases, which were studied one to one, in a simple way.
- If the spectrum is linear we can get the dynamical algebra.

Conclusions and remarks

- This method allow us to understand different cases, which were studied one to one, in a simple way.

- If the spectrum is linear we can get the dynamical algebra.

- The q-classical case. We have probed an analogous characterization theorem for the q-linear case, considering a special ς-commutator

$$[a(s), b(s)]_{\varsigma} = a(s)b(s) - \varsigma b(s)a(s),$$

and an analogous q-Lie algebra.

Conclusions and remarks

- ◇ This method allow us to understand different cases, which were studied one to one, in a simple way.
- ◇ If the spectrum is linear we can get the dynamical algebra.
- ◇ The q-classical case. We have probed an analogous characterization theorem for the q-linear case, considering a special ς-commutator

$$[a(s), b(s)]_{\varsigma} = a(s)b(s) - \varsigma b(s)a(s),$$

and an analogous q-Lie algebra.

- ► Continuous big q-Hermite functions.

Open problems

1. what is the dynamical in the non linear (resp. q-linear) case?.

Open problems

1. what is the dynamical in the non linear (resp. q-linear) case?.

2. Uniqueness and characterization.

Open problems

1. what is the dynamical in the non linear (resp. q-linear) case?.
2. Uniqueness and characterization.
3. The case Askey-Wilson and particular subcases.

Open problems

1. what is the dynamical in the non linear (resp. q-linear) case?.
2. Uniqueness and characterization.
3. The case Askey-Wilson and particular subcases.
4. Extension to the semiclassical case and others.

Open problems

1. what is the dynamical in the non linear (resp. q-linear) case?.
2. Uniqueness and characterization.
3. The case Askey-Wilson and particular subcases.
4. Extension to the semiclassical case and others.