Spring eastern sectional meeting

Orthogonality of the big -1 Jacobi polynomials for non-standard parameters

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April 1st, 2023. Online

Outline

The basics

The polynomials

1. The big q-Jacobi polynomials

$$P_n(x;a,b,c;q) = {}_3F_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} ; q, q \right)$$

2. These polynomials for standard parameters are orthogonal

$$\langle \mathbf{u}^{\mathrm{BqJ}}, pq \rangle := \int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} p(x)q(x)d_qx$$

3. They satisfy a three-term recurrence relation:

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 1, 2, ...$$

where $p_0(x) = 1$ and $p_1(x) = x - \beta_0$.

The Favard Theorem

Let $(p_n(x))$ be a polynomial sequence satisfying the recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 1, 2, \dots.$$

If $\gamma_n \neq 0$ for all n then there exists a moment linear functional $\mathscr{L}$ such that $(p_n(x))$ is orthogonal with respect to it.

The degenerate Favard Theorem

If there exists $N \in \mathbb{N}$ such that $\gamma_N = 0$, then there exists a two moment linear functionals $\mathscr{L}_1$, $\mathscr{L}_2$ such that $(p_n(x))$ is orthogonal with respect to the bilinear form:

$$\langle p, q \rangle = \mathscr{L}_1(pq) + \mathscr{L}_2\left(\left(\mathscr{T}^N p\right)\left(\mathscr{T}^N q\right)\right),$$

where $\mathscr{T}$ is certain lowering operator.

1. The coefficient of the recurrence relation:

$$\beta_n = 1 - A_n - C_n, \quad \gamma_n = A_{n-1} C_n,$$

where

$$A_n = \frac{(1 - aq^{n+1})(1 - abq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})},$$

and

$$C_n = -acq^{n+1} \frac{(1 - q^n)(1 - abc^{-1}q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

Reference

Costas-Santos, R. S. and Sanchez-Lara, J. F. Orthogonality of q-polynomials for non-standard parameters. *Journal of Approximation Theory* 163, no. 9(2011), 1246 —1268

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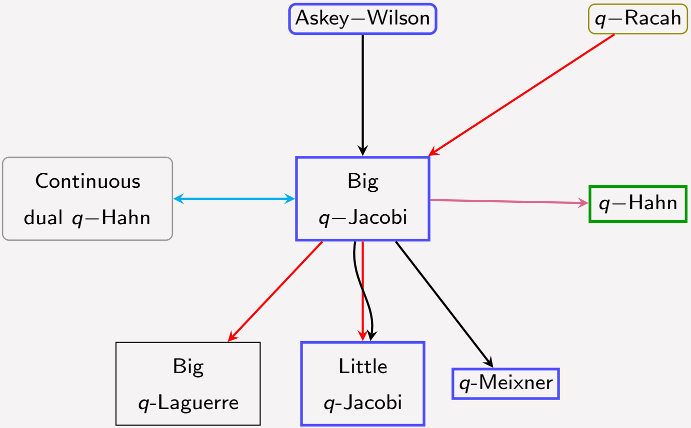

Askey−Wilson
q−Racah
Continuous
dual q−Hahn
Big
q−Jacobi
q−Hahn
Big
q-Laguerre
Little
q-Jacobi
q-Meixner

The limiting $q \to -1$ process

Reference

L. Vinet and A. Zhedanov, A ‘missing’ family of classical orthogonal polynomials, J. Phys. A 44 (2011), no. 8, 085201, 16.

Reference

L. Vinet and A. Zhedanov, A limit q = -1 for the big q-Jacobi polynomials, Trans. Amer. Math. Soc. 364 (2012), no. 10, 5491-5507.

- $q = -\exp(\epsilon)$
- $a = -\exp(\epsilon\alpha)$, $b = -\exp(\epsilon\beta)$, $c = -\exp(\epsilon c)$.
- We take $\epsilon \to 0$
- $$\lim_{\epsilon\to 0} \frac{(a;q)_k}{(a;q)_k} = \lim_{\epsilon\to 0} \frac{(-\exp(\epsilon\alpha); -\exp(\epsilon))_k}{(-\exp(\epsilon\beta); -\exp(\epsilon))_k}.$$

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The big -1 Jacobi polynomials

The Representation

- If n is even

$$Q_n^{(0)}(x;\alpha,\beta,c)=\kappa_n\left({}_2F_1\left(\begin{matrix}-\frac{n}{2},\frac{n+\alpha+\beta+2}{2}\\ \frac{\alpha+1}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right)\right.$$

$$\left.+\frac{n(1-x)}{(1+c)(\alpha+1)}\,{}_2F_1\left(\begin{matrix}1-\frac{n}{2},\frac{n+\alpha+\beta+2}{2}\\ \frac{\alpha+3}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right)\right)$$

- If n is odd

$$Q_n^{(0)}(x;\alpha,\beta,c)=\kappa_n\left({}_2F_1\left(\begin{matrix}-\frac{n-1}{2},\frac{n+\alpha+\beta+1}{2}\\ \frac{\alpha+1}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right)\right.$$

$$\left.-\frac{(\alpha+\beta+n+1)(1-x)}{(1+c)(\alpha+1)}\,{}_2F_1\left(\begin{matrix}-\frac{n-1}{2},\frac{n+\alpha+\beta+3}{2}\\ \frac{\alpha+3}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right)\right)$$

The coefficients of the recurrence relation

$$\beta_n^{(0)} = \begin{cases} -c + \dfrac{(c-1)n}{\alpha+\beta+2n} + \dfrac{(1+c)(\beta+n+1)}{\alpha+\beta+2n+2}, & n \text{ even} \\ c - \dfrac{(c-1)(n+1)}{\alpha+\beta+2n+2} - \dfrac{(1+c)(\beta+n)}{\alpha+\beta+2n}, & n \text{ odd} \end{cases}$$

$$\gamma_n^{(0)} = \begin{cases} \dfrac{(1-c)^2 n(\alpha+\beta+n)}{(\alpha+\beta+2n)^2}, & n \text{ even} \\ \dfrac{(1+c)^2(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}, & n \text{ odd} \end{cases}$$

The Factorization

The Gauss hypergeometric function fulfills the following factorization identity:

$$(-N+1)_{n+N}\,{}_2F_1\left(\begin{matrix}-n-N, a\\ -N+1\end{matrix};x\right) = (-N+1)_N\,{}_2F_1\left(\begin{matrix}-N, a\\ -N+1\end{matrix};x\right) \times (N+1)_n\,{}_2F_1\left(\begin{matrix}-n, a+N\\ N+1\end{matrix};x\right).$$

$$(-N+1)_N\,{}_2F_1\left(\begin{matrix}-N, a\\ -N+1\end{matrix};x\right) = (a)_N(-x)^N$$

The Factorization of the big -1 Jacobi polynomials

The Factorization of the big -1 Jacobi polynomials

For any $n, N \in \mathbb{N}$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$, with $\alpha = -2N-1$ or $\beta = -2N-1$, the following identities hold:

$$Q_{2N+1}^{(0)}(x; -2N-1, \beta, c) = (x^2-1)^N (x-1),$$

$$Q_{2N+1}^{(0)}(x; \alpha, -2N-1, c) = (x^2-c^2)^N (x+c),$$

$$Q_{2N+1+m}^{(0)}(x; -2N-1, \beta, c) = (-1)^m (x^2-1)^N (x-1) Q_m^{(0)}(-x; 2N+1, \beta, -c),$$

$$Q_{2N+1+m}^{(0)}(x; \alpha, -2N-1, c) = (x^2-c^2)^N (x+c) Q_m^{(0)}(x; \alpha, 2N+1, -c).$$

The property of orthogonality

Main theorem

For any $N \in \mathbb{N}_0$, $c \in \mathbb{C}$, $c \neq \pm 1$. The following statements hold:

- The polynomial sequence $\left(Q_n^{(0)}(x; -2N-1, \beta, c)\right)$ are orthogonal with respect to the bilinear form

$$\langle p, q \rangle_1 = \mathscr{L}_0(p, q) + \langle \mathbf{u}, (\tau_\alpha^{2N+1} p)(\tau_\alpha^{2N+1} q) \rangle.$$

- The polynomial sequence $\left(Q_n^{(0)}(x; \alpha, -2N-1, c)\right)$ are orthogonal with respect to the bilinear form

$$\langle p, q \rangle_2 = \mathscr{L}_1(p, q) + \langle \mathbf{u}, (\tau_\beta^{2N+1} p)(\tau_\beta^{2N+1} q) \rangle,$$

where the linear operator $\mathbf{u}$ is defined by

$$\langle \mathbf{u}, pq \rangle := \int_{[-c,-1]\cup[1,c]} p(-x; -c) q(-x; -c) \frac{x}{|x|} (x+1)(x^2-1)^{(\alpha-1)/2} (c-x)(c^2-x^2)^{(\beta-1)/2} \, dx,$$

Thank you!