Limit relations between q-Krall type orthogonal polynomials

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3 Our main purpose is to construct the q-Krall type orthogonal polynomials of the Hahn tableau. In this case, $\mathcal{C}_q$, is a q-classical linear functional.

1. Preliminaries

$\boxed{1}$ We start with q-classical linear functional

$$\langle \mathcal{C}_q, P\rangle = \sum_{s=a}^{b-1} P(x(s))\rho(s)\Delta x(s-\tfrac{1}{2}), \quad x(s) = q^{\pm s}.$$

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2 The q-polynomials satisfy a SODE, a TTRR

$$xP_n = \alpha_n P_{n+1} + \beta_n P_n + \gamma_n P_{n-1},$$

and the structure relations

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \widetilde{\alpha}_n P_{n+1}(s)_q + \widetilde{\beta}_n P_n(s)_q + \widetilde{\gamma}_n P_{n-1}(s)_q,$$

$$[\sigma(s) + \tau(s)\Delta x(s-\tfrac{1}{2}]\,\frac{\Delta P_n(s)_q}{\Delta x(s)} = \widehat{\alpha}_n P_{n+1}(s)_q + \widehat{\beta}_n P_n(s)_q + \widehat{\gamma}_n P_{n-1}(s)_q,$$

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3 The Christoffel-Darboux formula

$$K_n(s_1,s_2) := \sum_{m=0}^{n} \frac{P_m(s_1)_q P_m(s_2)_q}{d_m^2} = \frac{\alpha_n}{d_n^2} \frac{P_{n+1}(s_1)_q P_n(s_2)_q - P_{n+1}(s_2)_q P_n(s_1)_q}{x(s_1)-x(s_2)},$$

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$$K_{n-1}(s,s_0) = \frac{P_n(s_0)_q}{d_n^2 \left(\frac{\widetilde{\gamma}_n}{\gamma_n} - \frac{\widetilde{\alpha}_n}{\alpha_n} \right)} \left[\frac{\widetilde{\alpha}_n}{\alpha_n} P_n(s)_q - \frac{\sigma(s)}{x(s)-x(s_0)} \frac{\nabla P_n(s)_q}{\nabla x(s)} \right].$$

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[5] If $\sigma(s_0) + \tau(s_0)\nabla x_1(s_0) = 0$, then

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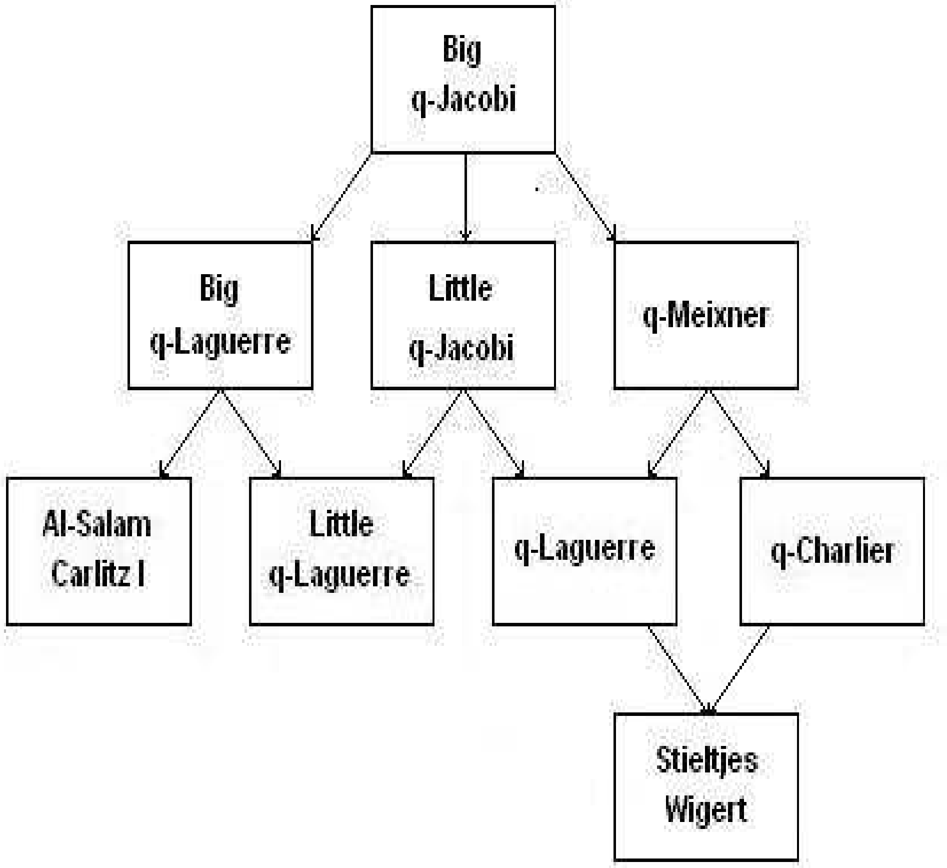
◇ $\mathbb{K}_n(s) := K_n(s,s)$.

2. The Hahn Tableau



3. The q-Krall type OP

¬ Let $\mathcal{C}$ be a q-classical linear functional ($d_0^2 = 1$)

$$\langle \mathcal{U}, P \rangle := \langle \mathcal{C}, P \rangle + AP(x_0) + BP(x_1), \quad A, B \geq 0,$$

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$$\widetilde{P}_n^{A,B}(s)_q := P_n(s)_q - A\widetilde{P}_n^{A,B}(x_0)_q K_{n-1}(x, x_0) - B\widetilde{P}_n^{A,B}(x_1)_q K_{n-1}(x, x_1).$$

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- Existence of The polynomials $P_n^{A,B}(s)_q$

$$\det \begin{bmatrix} 1 + A\mathbb{K}_{n-1}(x_0) & BK_{n-1}(x_0, x_1) \\ AK_{n-1}(x_1, x_0) & 1 + B\mathbb{K}_{n-1}(x_1) \end{bmatrix} \neq 0 \ \forall n \in \mathbb{N}.$$

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- One mass Case: $B = 0$

$$\widetilde{P}_n^{A}(s)_q = P_n(s)_q - A\widetilde{P}_n^{A}(x_0)_q Ker_{n-1}(x, x_0).$$

4. The Big q-Jacobi polynomials

- The q-classical linear functional

$$\langle \mathcal{C}^{BqJ}, P\rangle := \int_{cq}^{aq} P(x)\rho(x)d_qx,\ 0<a,b<q^{-1},\ c<0,$$

$$\rho(x)=\frac{1}{aq(1-q)}\frac{(aq,bq,cq,abc^{-1}q,a^{-1}x,c^{-1}x;q)_\infty}{(q,abq^2,a^{-1}c,ac^{-1}q,x,bc^{-1}x;q)_\infty},$$

$$d_n^2=\frac{(1-abq)(q,bq,abc^{-1}q;q)_n}{(1-abq^{2n+1})(aq,abq,cq;q)_n}\left(-acq^{\frac{n+3}{2}}\right)^n.$$

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◇ The mass points: $x_0 = aq$ and $x_1 = 1$.

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- We add 2 mass points at the end-points of the interval of orthogonality. $x_0 = aq$, $x_1 = cq$.

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$$\widetilde{P}_n^{A,B}(x;a,b,c;q) = (1-(1-q^{-n})A_n)P_n(x;a,b,c;q)+(1-q^{-1})B_n(x)\mathcal{D}_{q^{-1}}P_n(x;a,b,c;q).$$

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- We can write as basic hypergeometric series

$$\widetilde{P}_n^{A,B}(x;a,b,c;q) = \widetilde{D}_n(x)\, {}_5\varphi_4\left(\begin{array}{c} q^{-n},\ abq^{n+1},\ q^{1-\alpha_1},\ q^{1-\alpha_2},\ x \\ aq^2,\ cq^2,\ q^{-\alpha_1},\ q^{-\alpha_2} \end{array} \middle|\, q;q\right),$$

where

$$\widetilde{D}_n(x) = \frac{aq(1-(1-q^{-n})A_n)(1-q^{\alpha_1})(1-q^{\alpha_2})}{(1-aq)(1-cq)}\left[cq + \frac{bB_n(x)}{1-(1-q^{-n})A_n}\right].$$

$$(1-aq^{k+1})(1-cq^{k+1}) + \widetilde{B}_n(x)(1-abq^{n+k+1})(1-q^{-n+k}) = 0.$$

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7. One mass point cases

- Big q-Laguerre Krall can be represented as ${}_5\varphi_4$.
- Little q-Jacobi Krall can be represented as ${}_4\varphi_3$.
- q-Meixner-Krall can be represented as ${}_3\varphi_2$.
- Al-Salam Carlitz Krall I can be represented as ${}_4\varphi_3$.
- Little q-Laguerre Krall can be represented as ${}_3\varphi_2$.
- q-Charlier Krall can be represented as ${}_3\varphi_2$.
- Stieltjes-Wigert Krall can be represented as ${}_2\varphi_2$.

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Conclusions, remarks

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$**$ **THANKS FOR YOUR ATTENTION. ****