Classical orthogonal polynomials beyond the classical parameters

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Outline

THE BASICS

Classical Orthogonal Polynomials

- Let (P_n) be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1, \ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

- The weight function $d\mu(z) = \omega(z)\, dz$

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}, .$$

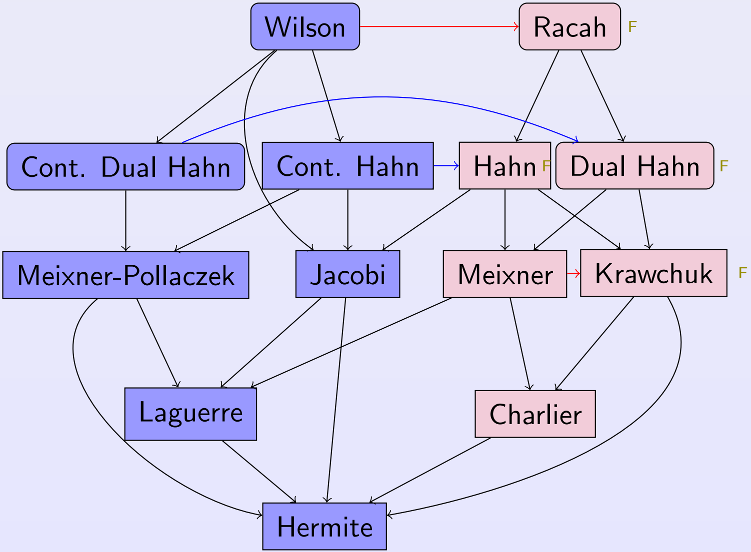
1. Continuous classical orthogonal polynomials
   - $\frac{d}{dx}(\phi(x)\omega(x)) = \psi(x)\omega(x),$
2. Δ-classical orthogonal polynomials
   - $\Delta(\phi(x)\omega(x)) = \psi(x)\omega(x),$
   - $\Delta f(x) = f(x+1) - f(x)$, $\nabla f(x) = f(x) - f(x-1),$
3. q-classical orthogonal polynomials
   - $\frac{\Delta(\phi(x)\omega(x))}{\Delta x(s-1/2)} = \psi(x)\omega(x),$
   - $\mathscr{D}_q f(x) = \frac{f(qx)-f(x)}{x(q-1)}$, $x \neq 0$, $\mathscr{D}_q f(0) = f'(0),$
   - $x(s) = c_1 q^s + c_2 q^{-s} + c_3.$

Some families

- Continuous Classical OP: Jacobi, Hermite, Laguerre and Bessel.
- Δ-Classical OP: Hahn, Racah, Meixner, Krawtchouk, Charlier, etc.
- q-Classical OP: Askey Wilson, q-Racah, q-Hahn, Continuous q-Hahn, Big q-Jacobi, q-Hermite, q-Laguerre, Al-Salam-Chihara, Stieltjes-Wigert, etc.

THE RELEVANT FAMILIES

The Classical Hypergeometric Orthogonal Polynomials



THE SCHEME IS TOO BIG TO PUT IT ON HERE.

SOME RESULTS

Characterization Theorems. The continuous version

Let (P_n) be an OPS with respect to ω. The following statements are equivalent:

1. P_n is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. (P'_{n+1}) is a OPS.
3. $(P^{(k)}_{n+k})$ is a OPS for any integer k.
4. (First structure relation)

$$\phi(x)P'_n(x) = \widehat{\alpha}_n P_{n+1}(x) + \widehat{\beta}_n P_n(x) + \widehat{\gamma}_n P_{n-1}(x).$$

5. (Second structure relation)

$$P_n(x) = \widetilde{\alpha}_n P'_{n+1}(x) + \widetilde{\beta}_n P'_n(x) + \widetilde{\gamma}_n P'_{n-1}(x).$$

6. (Eigenfunctions of SODE)

$$\phi(x)P''(x) + \psi(x)P'(x) + \lambda P(x) = 0.$$

Characterization Theorem (cont.)

Let (P_n) be an OPS with respect to ω. The following statements are equivalent:

1. P_n is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. The Rodrigues Formula for P_n

$$P_n(x) = \frac{B_n}{\omega(x)} \frac{d^n}{dx^n}(\phi^n(x)\omega(x)), \qquad B_n \neq 0.$$

Hypergeometric and basic hypergeometric representations

The continuous and discrete COP can be written in terms of

$$ {}_rF_s\left(\begin{array}{c} a_1,\ a_2,\ \ldots,a_r \\ b_1,\ b_2,\ \ldots,b_s \end{array} \middle| z \right) = \sum_{k\geq 0} \frac{(a_1)_k(a_2)_k\ldots(a_r)_k}{(b_1)_k(b_2)_k\ldots(b_s)_k}\frac{z^k}{k!}. $$

The q-discrete COP can be written in terms of

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\ \ldots,a_r \\ b_1,\ \ldots,b_s \end{array} \middle| z \right) = \sum_{k\geq 0} \frac{(a_1;q)_k\ldots(a_r;q)_k}{(b_1;q)_k\ldots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}. $$

$$ (a)_k = a(a+1)\cdots(a+k-1) $$

$$ (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) $$

The Favard's theorem

Let $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x).$$

Favard's theorem

If $\gamma_n \neq 0 \ \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with r_n a non-vanishing normalization factor.

Degenerate version of Favard's theorem

Theorem

If there exists N so that $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g \rangle = \mathscr{L}_0(fg) + \sum_{j \in \mathscr{A}} \mathscr{L}_1(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Here $\mathscr{T}$ is certain lower operator. In fact, ...

The operator $\mathscr{T}$

1. Among all the possible choices the linear operator $\mathscr{T}$ can be chosen as The "Associating operator"

$$\mathscr{T}(p)(x) = \mathscr{L}_0\left(\frac{p(x)-p(t)}{x-t}\right).$$

2.
   - If (p_n) is classical, then $\mathscr{T}$ is
   - the derivative, or
   - a difference operator.
3. And now ... the examples.

THE EXAMPLES

The monic ones are $p_n(x; a, b, c, d; q) \equiv p_n(x)$

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

with

$$\frac{\gamma_n}{1-q^n} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{4(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$$

Case $abcd \in \{q^{-k} : k \in \mathbb{N}_0\}$ are not considered since they are not normal.
They are symmetric with respect to any rearrangement of the parameters a, b, c, d.

$$\{n \in \mathbb{N} : \gamma_n = 0\} \neq \emptyset \iff ab, ac, \ldots, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$$

$$\iff \text{ they are } q\text{-Racah.}$$

Orthogonality of AW polynomials for $|q| < 1$

$$\int_C p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n\delta_{n,m}$$

where

- W is analytic in $\mathbb{C}$ except at the poles 0,

$$aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0 \quad \text{(the convergent poles)}$$

$$(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0 \quad \text{(the divergent poles)}$$

- C is the unit circle deformed to separate the convergent form the divergent poles.

The 3 key cases

- Case I: $a^2 = q^{-N+1}$ and

$$b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case II: $ab = q^{-N+1}$ and

$$a^2, b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case III: $ab = q^{-N+1}$, $a^2 = q^{-M}$ with $M \in \{0, 1, \ldots, N-2\}$ and

$$b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1$: By using the identity

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

- $|q| = 1$: If $q = \exp(2M\pi/N \text{ I})$, then $\gamma_{jN} = 0, j \in \mathbb{N}$.

  - Spiridonov and Zhedanov found $\mathscr{L}_0$
  - For $n > N$

$$\mathscr{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q).$$

  - $\mathscr{L}_j(p(\cdot)) = \mathscr{L}_0(p((-1)^M \cdot))$

- For the rest of the values of q the result keeps unknown.

The First example: Al-Salam-Carlitz Polynomials

The Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ are given explicitly by

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} \frac{q^k x^k}{a^k},$$

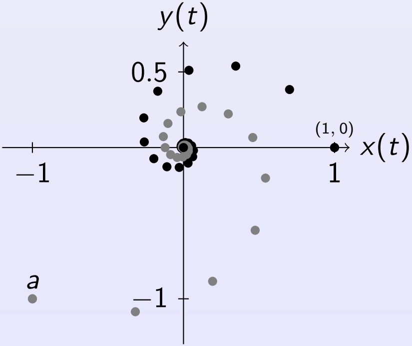
Result:
For any $a \in \mathbb{C}$, $a \neq 0,1$, and for any $q \in \mathbb{C}$, $0 < |q| < 1$, the Al-Salam-Carlitz polynomials are the unique (up to a multiplicative constant) satisfying the property of orthogonality

$$\sum_{k=0}^{\infty} \omega(q^k;a;q) U_m^{(a)}(q^k;q) U_n^{(a)}(q^k;q) q^k - \sum_{k=0}^{\infty} \omega(aq^k;a;q) U_m^{(a)}(aq^k;q) U_n^{(a)}(aq^k;q) aq^k = d_n^2 \delta_{nm}.$$

Here $\omega(x;a;q) = (qx;q)_\infty (qx/a;q)_\infty$.

What about the $a = 1$ case?

And, what about the $|q| = 1$ case?



The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(-1-i)q^k : k \in \mathbb{N}_0\}$ with $q = 0.8\exp(i\pi/6)$

The another example: The q-Laguerre polynomials

The (monic) q-Laguerre polynomials can be written as

$$L_n^{(\alpha)}(x;q) = (-1)^n q^{-n(n+\alpha)} {}_2\phi_1\left(\begin{array}{c} q^{-n}, -x \\ 0 \end{array}; q, q^{n+\alpha+1}\right), \quad \alpha > -1.$$

They fulfill the three-term recurrence relation

$$xL_n^{(\alpha)}(x;q) = L_{n+1}^{(\alpha)}(x;q) + \beta_n L_n^{(\alpha)}(x;q) + \gamma_n L_{n-1}^{(\alpha)}(x;q),$$

with

$$\gamma_n = q^{-4n-2\alpha+1}(1-q^n)(1-q^{n+\alpha})$$

The $-2 < \alpha < -1$ case

Result:

The polynomial sequence $(L_n^{(\alpha)}(x;q))$, with $-2 < \alpha < -1$ is the unique monic orthogonal polynomial sequence with respect to the linear form

$$\langle \mathbf{u}_\alpha, f \rangle := \int_0^\infty \big(f(x) - f(0)\big) \frac{x^\alpha}{(-x;q)_\infty} \, d_q x - M_\alpha f(0), \qquad f \in \mathbb{P},$$

where

$$M_\alpha = \frac{q-1}{2} \frac{(q, -q^{-\alpha+1}, -q^{-\alpha}; q)_\infty}{(q^{\alpha+1}, -q, -q; q)_\infty} > 0.$$

Of course, this result can be extended to the whole real line and even further... this is under construction.

Some References

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Thanks....

THANK YOU

FOR YOUR ATTENTION !!