

Zeros of polynomials orthogonal with respect to a signed weight

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Abstract

In this paper we consider the polynomial sequence $(P_n^{\alpha,q}(x))$ that is orthogonal on $[-1, 1]$ with respect to the weight function $x^{2q+1}(1-x^2)^\alpha(1-x)$, $\alpha > -1$, $q \in \mathbb{N}$; we obtain the coefficients of the three-term recurrence relation (TTRR) by using a different method from the one derived in [2]; we prove that the interlacing property does not hold properly for $(P_n^{\alpha,q}(x))$; and we also prove that, if $x_{n,n}^{\alpha+i,q+j}$ is the largest zero of $P_n^{\alpha+i,q+j}(x)$, $x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}$, $0 \leq i < j \leq n-1$.

Key words: Zeros, Real-rooted polynomials, Generalized Gegenbauer polynomials

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1. Introduction

It is a well known fact that if (p_n) is orthogonal with respect to a (real) weight function, namely $w(x)$, and such weight function is positive on $[a, b]$, then the zeros of p_n are real, distinct, interlace, and lie inside $]a, b[$, but such interlacing property is no longer valid when the weight is a signed function.

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In fact, Perron [9] proved that when the $w(x)$ changes sign once then one of zeros can lie outside of $[a, b]$.

In this paper we prove that such zero can lie into one of the endpoints of the interval, a or b . We consider the weight function $w(x) = x^{2q+1}(1 - x^2)^\alpha(1 - x)$, $\alpha > -1$, $q \in \mathbb{N}$ that changes sign once, at $x = 0$, and we prove that all the zeros are real, non interlacing, and that one of the zeros is the endpoint $a = -1$.

A sequence of monic orthogonal polynomials (\hat{Q}_n) satisfies for $n \geq 0$ the following TTRR [4]:

$$\hat{Q}_{n+2}(x) = (x - \beta_{n+1})\hat{Q}_{n+1}(x) - \gamma_{n+1}\hat{Q}_n(x), \quad (1)$$

with initial conditions $\hat{Q}_0(x) = 1$, $\hat{Q}_1(x) = x - \beta_0$, being (β_n) and (γ_n) the coefficients of the recurrence relation. The recurrence coefficients of $(P_n^{\alpha,q})$ were calculated in [2] by using the Laguerre-Freud equations, and, later on, an explicit expression for $P_n^{\alpha,q}(x)$ was given in [1]. The main aim of this paper is to keep studying these polynomials, more precisely, the behavior of the zeros of $(P_n^{\alpha,q}(x))$.

People working on zeros of orthogonal polynomials know how difficult is to explore this area. In fact, even in the case of Jacobi polynomials results on zeros are presented as conjectures (see [5], [6]).

In order to do this study, we use generalized Gegenbauer polynomials $(GG_n^{\alpha,\mu})$ that are orthogonal on $[-1, 1]$ with respect to the weight function $|x|^\mu(1 - x^2)^\alpha$, $\alpha > -1$, $\mu > -1$. Some properties of GG-polynomials can be found in [10] and [3].

The structure of the paper is the following: in Section 2 we present basic definitions, some notation, and a few preliminary results, in Section 3 we obtain some algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials as well as the recurrence coefficients of the TTRR fulfilled by $(P_n^{\alpha,q}(x))$, and in Section 4 some results regarding zeros of $(P_n^{\alpha,q}(x))$ are given.

2. Basic definitions and preliminary results

The *Pochhammer symbol*, or shifted factorial, is defined as

$$(\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n \geq 1. \quad (2)$$

The *Gauss's hypergeometric function* is

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} z^n, \quad \alpha, \beta \in \mathbb{C}; \gamma \in \mathbb{C} \setminus \mathbb{Z}_-; |z| < 1. \quad (3)$$

When α , or β , is a negative integer the hypergeometric serie (3) terminates, i.e. it reduces to a polynomial (of degree α , or β resp.).

Observe that after straightforward calculation one gets

$${}_2F_1(-m, \beta; \gamma; z) = \sum_{n=0}^m \frac{(-m)_n (\beta)_n}{n! (\gamma)_n} z^n = \sum_{n=0}^m \binom{m}{n} \frac{(\beta)_n}{(\gamma)_n} (-z)^n. \quad (4)$$

Denoting by

$${}_2F_1(\alpha, \beta; \gamma; z) \equiv F, \quad {}_2F_1(\alpha \pm 1, \beta; \gamma; z) \equiv F(\alpha \pm 1),$$

$${}_2F_1(\alpha, \beta \pm 1; \gamma; z) \equiv F(\beta \pm 1), \quad {}_2F_1(\alpha, \beta; \gamma \pm 1; z) \equiv F(\gamma \pm 1).$$

The functions $F(\alpha \pm 1)$, $F(\beta \pm 1)$, and $F(\gamma \pm 1)$ are said to be *contiguous* of F [7, p. 242]. Among the relations of this type we cite the following ones:

$$(\gamma - \alpha - \beta)F + \alpha(1 - z)F(\alpha + 1) - (\gamma - \beta)F(\beta - 1) = 0, \quad (5)$$

$$(\gamma - \alpha - 1)F + \alpha F(\alpha + 1) - (\gamma - 1)F(\gamma - 1) = 0, \quad (6)$$

$$\gamma(1 - z)F - \gamma F(\alpha - 1) + (\gamma - \beta)zF(\gamma + 1) = 0, \quad (7)$$

$$(\alpha - \beta)F - \alpha F(\alpha + 1) + \beta F(\beta + 1) = 0, \quad (8)$$

$$(\alpha - \beta)(1 - z)F + (\gamma - \alpha)F(\alpha - 1) - (\gamma - \beta)F(\beta - 1) = 0. \quad (9)$$

The following result allow us to find new relations between the the polynomial sequence $(P_n^{\alpha, q}(x))$ and the GG-polynomials:

Proposition 2.1. *GG polynomials are related by the following relation*

$$GG_{2n+1}^{\alpha, \mu}(x) = x GG_{2n}^{\alpha, \mu+2}(x), \quad n \geq 0. \quad (10)$$

Proof: This comes directly from

$$GG_{2n}^{\alpha, \mu}(x) = x^{2n} {}_2F_1\left(-n, -n - \frac{\mu}{2} + \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} + \frac{1}{2}; \frac{1}{x^2}\right), \quad n \geq 0, \quad (11)$$

and

$$GG_{2n+1}^{\alpha, \mu}(x) = x^{2n+1} {}_2F_1\left(-n, -n - \frac{\mu}{2} - \frac{1}{2}; -2n - \alpha - \frac{\mu}{2} - \frac{1}{2}; \frac{1}{x^2}\right), \quad n \geq 0. \quad (12)$$

□

The following result characterizes the polynomial sequence, up to a constant, through its property of orthogonality:

Theorem 2.2. [7] Let (p_n) be a sequence of polynomials. The following statements are equivalent:

(a) (p_n) is a orthogonal polynomial sequence with respect to the weight $w(x)$ on $[a, b]$.

(b) $\int_a^b w(x)\pi(x)p_n(x)dx = 0$ for every polynomial π of degree $m < n$

and $\int_a^b w(x)\pi(x)p_n(x)dx \neq 0$ if $m = n$.

(c) $\int_a^b w(x)x^m p_n(x)dx = k_n \delta_{m,n}$ with $k_n \neq 0$, $0 \leq m \leq n$.

Corollary 2.3. Let (p_n) be a orthogonal polynomial sequence with respect to the weight $w(x)$ on $[a, b]$ and let (Q_n) be another polynomial sequence that fulfills the following property of orthogonality:

$$\begin{cases} \int_a^b w(x)x^k Q_{n+r}(x)dx = 0, & 0 \leq k \leq n-1, \\ \int_a^b w(x)x^n Q_{n+r}(x)dx \neq 0, & n \geq 0, \end{cases} \quad (13)$$

then

$$Q_{n+r}(x) = \sum_{i=n}^{n+r} \lambda_i p_i(x), \quad (\lambda_i) \in \mathbb{C}, \quad n, r \in \mathbb{N}. \quad (14)$$

3. Algebraic relations between $(P_n^{\alpha,q}(x))$ and the GG-polynomials

Proposition 3.1. For any $n \geq 0$, and any integer q the following identities hold:

$$P_{2n}^{\alpha,q}(x) = GG_{2n}^{\alpha,2q+2}(x), \quad (15)$$

$$P_{2n+1}^{\alpha,q}(x) = (1+x)GG_{2n}^{\alpha+1,2q+2}(x). \quad (16)$$

Remark 3.1. Observe that with (15), we can write the last equation as

$$P_{2n+1}^{\alpha,q}(x) = (1+x)P_{2n}^{\alpha+1,q}(x), \quad (17)$$

one should point out that we have α in the left hand side whereas we have $\alpha + 1$ in the right hand side.

Proof: Let us start with the first identity. For $n \geq 1$, we get

$$\int_{-1}^1 |x|^\mu (1-x^2)^\alpha x^k GG_{2n}^{\alpha, \mu}(x) dx = 0, \quad 0 \leq k \leq 2n-1,$$

taking $\mu = 2q+2$, for $1 \leq k \leq 2n-1$ we get

$$\int_{-1}^1 x^{2q+1} (1-x^2)^\alpha (1-x)x^k GG_{2n}^{\alpha, 2q+2}(x) dx = 0.$$

Moreover, if $k = 0$ a direct calculation shows

$$\int_{-1}^1 x^{2q+1} (1-x^2)^\alpha (1-x) GG_{2n}^{\alpha, 2q+2}(x) dx = 0,$$

since it can be split in two parts and by using Proposition 2.1, and the property of orthogonality of $(GG_n^{\alpha, \mu})$ the previous integral vanishes.

By analogous reasons we also obtain

$$\begin{aligned} \int_{-1}^1 x^{2q+1} (1-x^2)^\alpha (1-x)x^{2n} GG_{2n}^{\alpha, 2q+2}(x) dx = \\ - \int_{-1}^1 x^{2q+2} (1-x^2)^\alpha x^{2n} GG_{2n}^{\alpha, 2q+2}(x) dx \neq 0. \end{aligned}$$

Hence, by Theorem 2.2, the first identity holds.

Remark 3.2. By using Eq. (11) we get for $n \geq 0$ the hypergeometric representation for $P_{2n}^{\alpha, q}(x)$:

$$P_{2n}^{\alpha, q}(x) = x^{2n} {}_2F_1\left(-n, -n-q-\frac{1}{2}; -2n-\alpha-q-\frac{1}{2}; \frac{1}{x^2}\right), \quad (18)$$

that we can be written as [8, V1 p. 40 (23)]

$$P_{2n}^{\alpha, q}(x) = \frac{(-1)^n (q + \frac{3}{2})_n}{(n + q + \alpha + \frac{3}{2})_n} {}_2F_1\left(-n, n+q+\alpha + \frac{3}{2}; q + \frac{3}{2}; x^2\right). \quad (19)$$

Next let us prove the second identity. For $n \geq 1$, we know

$$\begin{aligned} \int_{-1}^1 |x|^\mu (1-x^2)^\alpha x^k GG_{2n}^{\alpha, \mu}(x) dx = 0, \\ \int_{-1}^1 |x|^\mu (1-x^2)^\alpha x^{2n} GG_{2n}^{\alpha, \mu}(x) dx \neq 0. \end{aligned}$$

So, setting $\alpha \leftarrow \alpha + 1$, $\mu \leftarrow 2q + 2$, $0 \leq k \leq 2n - 1$, we get

$$\int_{-1}^1 x^{2q+2}(1-x^2)^{\alpha+1}x^k GG_{2n}^{\alpha+1,2q+2}(x)dx = 0,$$

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)x^{k+1}\left((1+x)GG_{2n}^{\alpha+1,2q+2}(x)\right)dx = 0.$$

So, for $1 \leq k \leq 2n$, we have

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)x^k\left((1+x)GG_{2n}^{\alpha+1,2q+2}(x)\right)dx = 0,$$

and if $k = 0$, by parity of $GG_{2n}^{\alpha+1,2q+2}(x)$, we obtain

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)\left((1+x)GG_{2n}^{\alpha+1,2q+2}(x)\right)dx = 0.$$

Therefore, since $\deg((1+x)GG_{2n}^{\alpha+1,2q+2}(x)) = 2n + 1$ and

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)x^{2n+1}\left((1+x)GG_{2n}^{\alpha+1,2q+2}(x)\right)dx =$$

$$\int_{-1}^1 x^{2q+2}(1-x^2)^{\alpha+1}x^{2n}GG_{2n}^{\alpha+1,2q+2}(x)dx \neq 0.$$

Then, by Theorem 2.2, we have

$$(1+x)GG_{2n}^{\alpha+1,2q+2}(x) = P_{2n+1}^{\alpha,q}(x),$$

and hence the second identity holds. \square

Once we have got these algebraic relations we can compute the recurrences coefficients associated to the polynomial sequence $(P_n^{\alpha,q}(x))$.

Remark 3.3. Notice that due the expression of the integrals and the weight functions we can find a link between the polynomials $(P_n^{\alpha,q})$ and GG -polynomials, which was not possible to do with Laguerre-Freud equation [2] nor with the explicit representation of $P_n^{\alpha,q}(x)$ [1].

Proposition 3.2. The monic polynomial sequence $(P_n^{\alpha,q}(x))$ fulfills for $n \geq 0$ the following TTRR:

$$P_{n+2}^{\alpha,q}(x) = (x - \beta_{n+1}^{\alpha,q})P_{n+1}^{\alpha,q}(x) - \gamma_{n+1}^{\alpha,q}P_n^{\alpha,q}(x), \quad (20)$$

with initial conditions $P_0^{\alpha,q}(x) = 1$, $P_1^{\alpha,q}(x) = x - \beta_0^{\alpha,q}$; where

$$\beta_n^{\alpha,q} = (-1)^{n+1}, \quad (21)$$

$$\gamma_{2n}^{\alpha,q} = -2 \frac{n(2n+2q+1)}{(4n+2\alpha+2q+1)(4n+2\alpha+2q+3)}, \quad (22)$$

$$\gamma_{2n+1}^{\alpha,q} = -2 \frac{(n+\alpha+1)(2n+2\alpha+2q+3)}{(4n+2\alpha+2q+3)(4n+2\alpha+2q+5)}. \quad (23)$$

Proof: For all $n \geq 0$ we have

$$\beta_n^{\alpha,q} = \frac{\int_{-1}^1 (x^{2q+1}(1-x^2)^\alpha(1-x))x(P_n^{\alpha,q}(x))^2 dx}{\int_{-1}^1 (x^{2q+1}(1-x^2)^\alpha(1-x))(P_n^{\alpha,q}(x))^2 dx}.$$

So, setting $2n \leftarrow n$ one gets

$$\beta_{2n}^{\alpha,q} = \frac{\int_{-1}^1 x^{2q+2}(1-x^2)^\alpha(1-x)(P_{2n}^{\alpha,q}(x))^2 dx}{\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha(1-x)(P_{2n}^{\alpha,q}(x))^2 dx},$$

since $P_{2n}^{\alpha,q}$ is even we obtain

$$\beta_{2n}^{\alpha,q} = \frac{\int_{-1}^1 x^{2q+2}(1-x^2)^\alpha(P_{2n}^{\alpha,q}(x))^2(x)dx}{-\int_{-1}^1 x^{2q+2}(1-x^2)^\alpha(P_{2n}^{\alpha,q}(x))^2 dx} = -1.$$

The odd case is completely analogous and it will be omitted.

In order to obtain $\gamma_{n+1}^{\alpha,q}$ we use (20) together with $\beta_n^{\alpha,q} = (-1)^{n+1}$, i.e. since for $n \geq 0$ we get

$$P_{n+2}^{\alpha,q}(x) = (x - (-1)^n)P_{n+1}^{\alpha,q}(x) - \gamma_{n+1}^{\alpha,q}P_n^{\alpha,q}(x),$$

then, taking into account Eq. (17), we deduce

$$\gamma_{2n}^{\alpha,q} = \frac{P_{2n}^{\alpha,q}(x) - P_{2n}^{\alpha+1,q}(x)}{P_{2n-2}^{\alpha+1,q}(x)},$$

$$\gamma_{2n+1}^{\alpha,q} = \frac{(x^2 - 1)P_{2n}^{\alpha+1,q}(x) - P_{2n+2}^{\alpha,q}(x)}{P_{2n}^{\alpha,q}(x)}.$$

To compute $\gamma_{2n}^{\alpha,q}$ we use (19) and (8) with $a = n + q + \alpha + \frac{3}{2}$, $b = -n$, $c = q + \frac{3}{2}$, and $t = x^2$, obtaining

$$\gamma_{2n}^{\alpha,q} = \frac{(-1)^n \frac{(c)_n}{(a)_n} F - (-1)^n \frac{(c)_n}{(a+1)_n} F(a+1)}{(-1)^{n-1} \frac{(c)_{n-1}}{(a)_{n-1}} F(b+1)},$$

and since $(\lambda + 1)_k = \frac{\lambda+k}{\lambda}(\lambda)_k$ we get

$$\begin{aligned}\gamma_{2n}^{\alpha,q} &= -\frac{(c)_n(a)_{n-1}}{(c)_{n-1}(a)_n} \frac{F - \frac{a}{a+n}F(a+1)}{F(b+1)} \\ &= -\frac{(c)_n(a)_{n-1}}{(a-b)(c)_{n-1}(a)_n} \frac{(a-b)F - aF(a+1)}{F(b+1)} = b \frac{(c)_n(a)_{n-1}}{(a-b)(c)_{n-1}(a)_n},\end{aligned}$$

so

$$\gamma_{2n}^{\alpha,q} = b \frac{(c)_n(a)_{n-1}}{(a-b)(c)_{n-1}(a)_n}$$

thus

$$\begin{aligned}\gamma_{2n}^{\alpha,q} &= -n \frac{(q + \frac{3}{2})_n (n + q + \alpha + \frac{3}{2})_{n-1}}{(2n + q + \alpha + \frac{3}{2})(q + \frac{3}{2})_{n-1} (n + q + \alpha + \frac{3}{2})_n} \\ &= -n \frac{(q + \frac{3}{2} + n - 1)}{(2n + q + \alpha + \frac{3}{2})(2n + q + \alpha + \frac{3}{2} - 1)} \\ &= -2 \frac{n(2n + 2q + 1)}{(4n + 2q + 2\alpha + 1)(4n + 2q + 2\alpha + 3)}.\end{aligned}$$

To compute γ_{2n+1} we use (19) and (9) with $a = n + q + \alpha + \frac{5}{2}$, $b = -n$, $c = q + \frac{3}{2}$, and $t = x^2$, obtaining

$$\gamma_{2n+1}^{\alpha,q} = \frac{(-1)^n \frac{(c)_n}{(a)_n} (t-1)F - (-1)^{n+1} \frac{(c)_{n+1}}{(a)_{n+1}} F(b-1)}{(-1)^n \frac{(c)_n}{(a-1)_n} F(a-1)}.$$

Since $(c)_{n+1} = (c+n)(c)_n$ and $(a)_{n+1} = (a+n)(a)_n$ we have

$$\gamma_{2n+1}^{\alpha,q} = \frac{(a-1)_n}{(a+n)(a)_n} \frac{(t-1)(a-b)F + (c-b)F(b-1)}{F(a-1)} = (c-a) \frac{(a-1)_n}{(a+n)(a)_n},$$

thus

$$\begin{aligned}\gamma_{2n+1}^{\alpha,q} &= (q + \frac{3}{2} - (n + q + \alpha + \frac{5}{2})) \frac{(n + q + \alpha + \frac{5}{2} - 1)_n}{(2n + q + \alpha + \frac{5}{2})(n + q + \alpha + \frac{5}{2})_n} \\ &= -2 \frac{(n + \alpha + 1)(2n + 2\alpha + 2q + 3)}{(4n + 2\alpha + 2q + 3)(4n + 2\alpha + 2q + 5)}.\end{aligned}$$

□

4. Zeros of ($P_n^{\alpha,q}$)

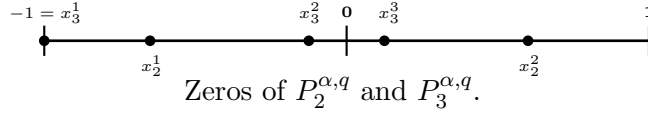
Using (10), (15) and (16) we can state the following result:

Theorem 4.1. *The following statements hold:*

- All the zeros of $P_{2n}^{\alpha,q}(x)$ are real.
- The Perron's zero is -1 .
- The zeros of $P_{2n}^{\alpha,q}(x)$ and the zeros of $P_{2n+1}^{\alpha,q}(x)$ do not interlace.

Proof: By (10) and (15), the first two statements follow. To prove the third one, it is sufficient to see that the zeros of $GG_{2n}^{\alpha,\mu}(x)$ and $GG_{2n}^{\alpha+1,\mu}(x)$ do not interlace. But, for all $x \in [-1, 1]$ we know that $GG_{2n}^{\alpha,\mu}(-x) = GG_{2n}^{\alpha,\mu}(x)$ thus $GG_{2n}^{\alpha,2q+2}$ and $GG_{2n}^{\alpha+1,2q+2}$ have exactly n zeros in $]0, 1[$.

Let $(x_{2n,k}^{\alpha,2q+2})_{1 \leq k \leq 2n}$ and $(x_{2n,k}^{\alpha+1,2q+2})_{1 \leq k \leq 2n}$ be the zeros of $GG_{2n}^{\alpha,2q+2}$ and $GG_{2n}^{\alpha+1,2q+2}$ respectively in increasing order. Then, between $-x_0$ and x_0 –that are consecutive zeros of $GG_{2n}^{\alpha,2q+2}(x)$ (resp. of $GG_{2n}^{\alpha+1,2q+2}$)– we can not find a zero of $GG_{2n}^{\alpha+1,2q+2}$ (resp. of $GG_{2n}^{\alpha,2q+2}$).



□

The following result allow us to obtain even more information regarding the zeros of ($P_{2n}^{\alpha,q}(x)$):

Proposition 4.2. *For $n \geq 0$, it holds:*

$$\frac{d}{dx} P_{2n}^{\alpha,q}(x) = 2n GG_{2n-1}^{\alpha+1,2q+2}(x). \quad (24)$$

Taking into account (17) and (10), we can write the last equality as

$$\frac{d}{dx} P_{2n}^{\alpha,q}(x) = 2nx P_{2n-2}^{\alpha+1,q+1}(x). \quad (25)$$

We also have

$$\frac{d}{dx} P_{2n+1}^{\alpha,q}(x) = P_{2n}^{\alpha+1,q}(x) + 2nx P_{2n-1}^{\alpha+1,q+1}(x). \quad (26)$$

Proof: Let us start proving (24). To do this it is enough to prove

$$\int_{-1}^1 x^{2q+2}(1-x^2)^{\alpha+1}x^k \left(P_{2n}^{\alpha,q}(x)\right)' dx = 0, \quad 0 \leq k \leq 2n-2.$$

But, integrating by parts once one gets

$$\begin{aligned} \int_{-1}^1 x^{2q+k+2}(1-x^2)^{\alpha+1} \left(P_{2n}^{\alpha,q}(x)\right)' dx &= - \int_{-1}^1 \left((2q+k+2)x^{2q+k+1}(1-x^2)^{\alpha+1} \right. \\ &\quad \left. - 2(\alpha+1)x^{2q+k+3}(1-x^2)^\alpha \right) P_{2n}^{\alpha,q}(x) dx \\ &= \int_{-1}^1 x^{2q+1}(1-x^2)^\alpha \left(2(\alpha+1)x^2 - (2q+k+2)(1-x^2) \right) x^k P_{2n}^{\alpha,q}(x) dx. \end{aligned}$$

If $k = 2p$, $0 \leq p \leq n-1$ then

$$\int_{-1}^1 x^{2q+1}(1-x^2)^\alpha \left((2\alpha+2q+2p+4)x^2 - (2q+2p+2) \right) x^{2p} P_{2n}^{\alpha,q}(x) dx = 0,$$

since we have an odd function in $[-1, 1]$.

If $k = 2p+1$, $0 \leq p \leq n-2$ then

$$\begin{aligned} \int_{-1}^1 x^{2q+1}(1-x^2)^\alpha \left(2(\alpha+1)x^2 - (2q+2p+3)(1-x^2) \right) x^{2p+1} P_{2n}^{\alpha,q}(x) dx \\ = 2(\alpha+1) \int_{-1}^1 x^{2q+1}(1-x^2)^\alpha x^{2p+3} P_{2n}^{\alpha,q}(x) dx \\ - (2q+2p+3) \int_{-1}^1 x^{2q+1}(1-x^2)^\alpha (1-x) \left(x^{2p+1}(1+x) \right) P_{2n}^{\alpha,q}(x) dx, \end{aligned}$$

where the second integral vanishes since the property of orthogonality holds and $2p+2 < 2n$, and the first integral can be written as

$$2(\alpha+1) \int_{-1}^1 x^{2q+1}(1-x^2)^\alpha (1-x) x^{2p+3} P_{2n}^{\alpha,q}(x) dx, \quad 0 \leq p \leq n-2,$$

that also vanishes because $2p+3 \leq 2n-1$.

Therefore

$$\int_{-1}^1 x^{2q+2}(1-x^2)^{\alpha+1}x^k \left(P_{2n}^{\alpha,q}(x)\right)' dx = 0, \quad 0 \leq k \leq 2n-2.$$

But $\deg \left(P_{2n}^{\alpha,q}(x) \right)' = 2n - 1$, so

$$\frac{d}{dx} P_{2n}^{\alpha,q}(x) = 2n G G_{2n-1}^{\alpha+1,2q+2}(x), \quad n \geq 1.$$

By using Eq. (24) and recalling identity (10), then (25) holds. To prove (26), we need to replace $P_{2n+1}^{\alpha,q}$ by $(1+x)P_{2n}^{\alpha+1,q}$ getting

$$\frac{d}{dx} P_{2n+1}^{\alpha,q}(x) = \left((1+x)P_{2n}^{\alpha+1,q}(x) \right)' = (1+x) \left(P_{2n}^{\alpha+1,q}(x) \right)' + P_{2n}^{\alpha+1,q}(x),$$

taking into account (25) we have

$$\left(P_{2n+1}^{\alpha,q}(x) \right)' = (1+x) \left(2nx P_{2n-2}^{\alpha+2,q+1}(x) \right) + P_{2n}^{\alpha+1,q}(x),$$

and using $P_{2n-1}^{\alpha+1,q+1}(x) = (1+x)P_{2n-2}^{\alpha+2,q+1}(x)$ the identity holds. \square

Note that these relations help us to obtain more information related to the zeros of both, $(P_n^{\alpha,q})$ and $(P_n^{\alpha+i,q+j})$, for any $i, j \geq 0$. The following result show us how relevant is that relation:

Proposition 4.3. *If we denote by $x_{n,n}^{\alpha+i,q+j}$ the largest zero of $P_n^{\alpha+i,q+j}$, then*

$$x_{2n-2j,2n-2j}^{\alpha+j,q+j} < x_{2n-2i,2n-2i}^{\alpha+i,q+i}, \quad 0 \leq i < j \leq n-1. \quad (27)$$

Proof: To prove (27) we need to use (25):

$$\frac{d}{dx} P_{2n}^{\alpha,q}(x) = 2nx P_{2n-2}^{\alpha+1,q+1}(x), \quad n \geq 0.$$

We know that $P_{2n}^{\alpha,q}(x)$ and $P_{2n-2}^{\alpha+1,q+1}(x)$ have n and $(n-1)$ zeros in $]0, 1[$ respectively and between two consecutive zeros of $P_{2n}^{\alpha,q}(x)$ we find, exactly, one zero of $P_{2n-2}^{\alpha+1,q+1}(x)$ then the largest zero of $P_{2n-2}^{\alpha+1,q+1}(x)$ is located between two zeros of $P_{2n}^{\alpha,q}(x)$ thus

$$x_{2n-2,2n-2}^{\alpha+1,q+1} < x_{2n,2n}^{\alpha,q},$$

an analog idea leads to

$$x_{2n-4,2n-4}^{\alpha+2,q+2} < x_{2n-2,2n-2}^{\alpha+1,q+1},$$

and so on. Then, we can write

$$\dots < x_{2n-2j,2n-2j}^{\alpha+j,q+j} < \dots < x_{2n-2i,2n-2i}^{\alpha+i,q+i} < \dots < x_{2n-2,2n-2}^{\alpha+1,q+1} < x_{2n,2n}^{\alpha,q}.$$

Hence the result holds. \square

Remark 4.1. Using Proposition 3.4. and the relation

$$x_{2n+1,m}^{\alpha+k,q+l} = x_{2n,m-1}^{\alpha+k+1,q+l}; \quad k, l \in \mathbb{N}, 2 \leq m \leq 2n+1,$$

with $x_{2n+1,1}^{\alpha+k,q+l} = -1$, we obtain

$$x_{2n-2j+1,2n-2j+1}^{\alpha+j-1,q+j} < x_{2n-2i+1,2n-2i+1}^{\alpha+i-1,q+i}, \quad 0 \leq i < j \leq n.$$

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