THE ORTHOGONALITY OF AL-SALAM-CARLITZ POLYNOMIALS FOR COMPLEX PARAMETERS

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Abstract. In this contribution, we study the orthogonality conditions satisfied by Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ when the parameters a and q are not necessarily real nor 'classical'. We establish orthogonality on a simple contour in the complex plane which depends on the parameters. In all cases we show that the orthogonality conditions characterize the Al-Salam-Carlitz Polynomials $U_n^{(a)}(x;q)$ of degree n up to a constant factor. We also obtain a generalization of the unique generating function for these polynomials.

1. Introduction

The Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ were introduced by W. A. Al-Salam and L. Carlitz in [1] as follows:

$$U_n^{(a)}(x;q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} \frac{q^k x^k}{a^k}. \tag{1.1}$$

In fact, these polynomials have a Rodrigues-type formula [4, (3.24.10)]

$$U_n^{(a)}(x;q) = \frac{a^n q^{\binom{n}{2}}(1-q)^n}{q^n w(x;a;q)} (\mathscr{D}_{q^{-1}})^n [w(x;a;q)], \qquad w(x;a;q) := (qx;q)_\infty (qx/a;q)_\infty,$$

where the q-Pochhammer symbol is defined as

$$(z;q)_0 := 1, \quad (z;q)_n := \prod_{k=0}^{n-1}(1-zq^k),$$

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1-zq^k), \quad |z|<1,$$

and the q-derivative operator is defined by

$$(\mathscr{D}_q f)(z) := \begin{cases} \dfrac{f(qz)-f(z)}{(q-1)z} & \text{if } q \neq 1 \wedge z \neq 0, \\ f'(z) & \text{if } q = 1 \vee z = 0. \end{cases}$$

The expression (1.1) shows us that $U_n^{(a)}(x;q)$ is an analytic function for any complex value parameters a and q, and thus can be considered for general $a, q \in \mathbb{C} \setminus \{0\}$.

The classical Al-Salam-Carlitz polynomials correspond to parameters $a<0$ and $0<q<1$. For these parameters, the Al-Salam-Carlitz polynomials are orthogonal on $[a,1]$ with respect to the weight function w. More specifically, for $a<0$ and $0<q<1$ [4, (14.24.2)],

$$\int_a^1 U_n^{(a)}(x;q) U_m^{(a)}(x;q) (qx, qx/a;q)_\infty d_q x = (-a)^n (1-q)(q;q)_n (q;q)_\infty (a;q)_\infty (q/a;q)_\infty q^{\binom{n}{2}} \delta_{nm},$$

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where the q-Jackson integral [4, (1.15.7)] is defined as

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

where

$$\int_0^a f(x)d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

Taking into account the previous orthogonality relation, it is a direct result that if a and q are classical, all their zeros are simple and belong to the interval $[a,1]$, but this is no longer valid for general a and q. In this paper we show that for general a, q complex numbers, but excluding some special cases, the Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ may still be characterized by orthogonality relations. The case $a<0$ and $0<q<1$ or $0<aq<1$ and $q>1$ are classical, and standard orthogonality on some interval on the real line takes place. Note that this is the key for the study of many properties of Al-Salam-Carlitz polynomials I and II. Thus, our goal is to establish orthogonality conditions for most of the remaining cases. We believe that these new orthogonality conditions can be useful in the study of the zeros of Al-Salam-Carlitz polynomials. For general $a,q\in\mathbb{C}\setminus\{0\}$, the zeros are not confined into a real interval, but they distribute themselves in the complex plane as we can see in Figure 1.

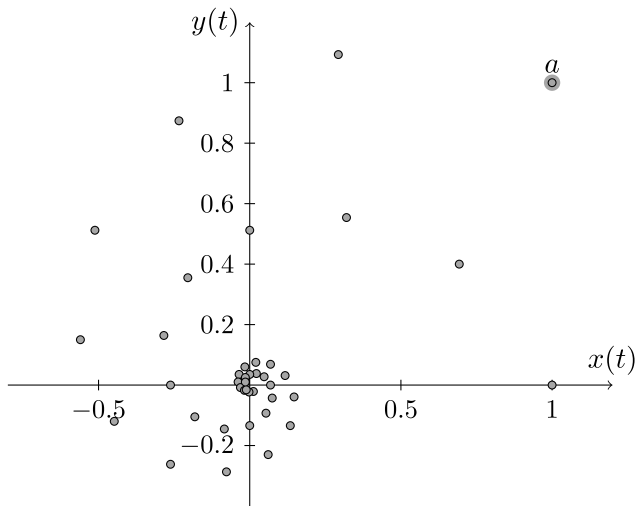

FIGURE 1. Zeros of $U_{30}^{(1+i)}\left(x;\frac{4}{5}\exp(\pi i/6)\right)$

2. Orthogonality in the complex plane

Theorem 2.1. *Let $a,q\in\mathbb{C}$, $a\neq 0,1$, $0<|q|<1$, the Al-Salam-Carlitz polynomials are the unique (up to a multiplicative constant) satisfying the property of orthogonality*

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)w(x;a;q)d_qx = d_n^2\delta_{nm}, \tag{2.1}$$

where $d_n^2=(-a)^n(1-q)(q;q)_n(q;p)_\infty(a;q)_\infty(q/a;q)_\infty q^{\binom{n}{2}}$.

Remark 2.2. Notice that if $0<|q|<1$, the lattice $\{q^k : k\in\mathbb{N}_0\}\cup\{aq^k : k\in\mathbb{N}_0\}$ is a set of points which are located inside of a single contour that goes from 1 to 0, and then from 0 to a, through two spirals which we can see in Figure 2. Taking into account (2.1), we need to avoid the $a=1$ case. For the $a=0$ case, we cannot apply Favard's result [2], because in such a case this polynomial sequence fulfills the recurrence relation [4]

$$U_{n+1}^{(0)}(x;q)=(x-q^n)U_n^{(0)}(x;q), \quad U_0^{(0)}(x;q)=1.$$

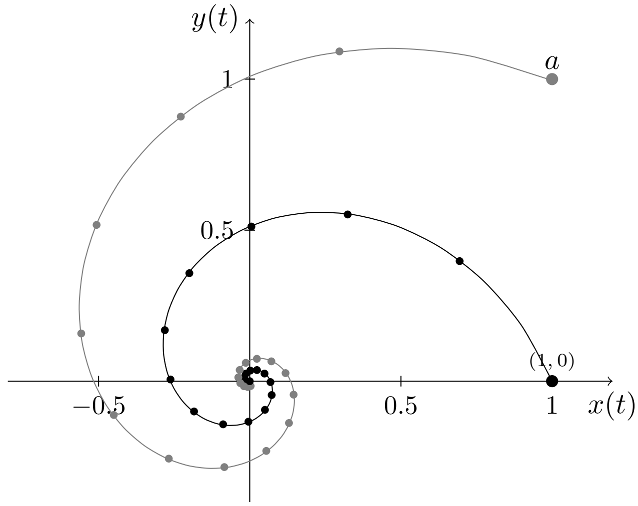

FIGURE 2. The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 4/5\exp(\pi i/6)$.

Proof. Let $0 < |q| < 1$, and $a \in \mathbb{C}$, $a \neq 0, 1$. We are going to express the q-Jackson integral (2.1) as the difference of the two infinite sums and apply the identity

$$\sum_{k=0}^{M} f(q^k)(\mathscr{D}_{q^{-1}}g)(q^k)q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1}-1} - \sum_{k=0}^{M} g(q^{k-1})(\mathscr{D}_{q^{-1}}f)(q^k)q^k. \tag{2.2}$$

Let $n \geq m$. Then, for one side since $w(q^{-1}; a; q) = 0$, and using the identities [4, (14.24.7), (14.24.9)], we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)w(q^k;a;q)q^k \\
&\quad = \frac{a(1-q)}{q^{2-n}} \lim_{M\to\infty} \sum_{k=0}^{M} \mathscr{D}_{q^{-1}}[w(q^k;a;q)U_{n-1}^{(a)}(q^k;q)]U_m^{(a)}(q^k;q)q^k \\
&\quad = aq^{n-1} \lim_{M\to\infty} U_m^{(a)}(q^M;q)U_{n-1}^{(a)}(q^M;q)w(q^M;a;q) \\
&\qquad + aq^{n-1}(q^m-1) \lim_{M\to\infty} \sum_{k=0}^{M-1} w(q^k;a;q)U_{n-1}^{(a)}(q^k;q)U_{m-1}^{(a)}(q^k;q)q^k.
\end{aligned}$$

Following an analogous process as before, and since $w(aq^{-1}; a; q) = 0$, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)w(aq^k;a;q)aq^k \\
&\quad = aq^{n-1} \lim_{M\to\infty} U_m^{(a)}(aq^M;q)U_{n-1}^{(a)}(aq^M;q)w(aq^M;a;q) \\
&\qquad + aq^{n-1}(q^m-1) \lim_{M\to\infty} \sum_{k=0}^{M-1} w(aq^k;a;q)U_{n-1}^{(a)}(aq^k;q)U_{m-1}^{(a)}(aq^k;q)aq^k.
\end{aligned}$$

Therefore, if $m < n$, and since m is finite one can first repeat the previous process $m+1$ times obtaining

$$\begin{aligned}
&\sum_{k=0}^{\infty} U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)w(q^k;a;q)q^k \\
&\quad = \lim_{M\to\infty} \sum_{\nu=1}^{m+1} (-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1};q)_\nu U_{m-\nu+1}^{(a)}(q^M;q)U_{n-\nu}^{(a)}(q^M;q)w(q^M;a;q),
\end{aligned}$$

and

$$\sum_{k=0}^{\infty} U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)w(aq^k;a;q)aq^k$$
$$= \lim_{M\to\infty}\sum_{\nu=1}^{m+1}(-aq^n)^\nu q^{-\nu(\nu+1)/2}(q^{-m+\nu-1};q)_\nu U_{m-\nu+1}^{(a)}(aq^M;q)U_{n-\nu}^{(a)}(aq^M;q)w(aq^M;a;q).$$

Hence since the difference of both limits, term by term, goes to 0 since $|q|<1$, then

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)(qx,qx/a;q)_\infty d_qx = 0.$$

For $n=m$, following the same idea, we have

$$\int_a^1 U_n^{(a)}(x;q)U_n^{(a)}(x;q)w(x;a;q)d_qx$$
$$= \frac{a(q^n-1)}{q^{1-n}}\sum_{k=0}^{\infty}\left(w(q^k;a;q)\left(U_{n-1}^{(a)}(q^k;q)\right)^2 q^k - aw(aq^k;a;q)\left(U_{n-1}^{(a)}(aq^k;q)\right)^2 q^k\right)$$
$$= (-a)^n(q;q)_n q^{\binom{n}{2}}\sum_{k=0}^{\infty}\left(w(q^k;a;q)q^k - a\, w(aq^k;a;q)q^k\right)$$
$$= (-a)^n(q;q)_n(q;q)_\infty\, q^{\binom{n}{2}}\sum_{k=0}^{\infty}\left((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty\right)\frac{q^k}{(q;q)_k}$$

Since it is known that in this case [4, (14.24.2)]

$$\int_a^1 U_n^{(a)}(x;q)U_n^{(a)}(x;q)w(x;a;q)d_qx = (-a)^n(q;q)_n(q;q)_\infty(a;q)_\infty(q/a;q)_\infty q^{\binom{n}{2}}.$$

Due to the normality of this polynomial sequence, i.e., $\deg U_n^{(a)}(x;q)=n$ for all $n\in\mathbb{N}_0$, the uniqueness is straightforward, hence the result holds. ☐

From this result, and taking into account that the squared norm for the Al-Salam-Carlitz polynomials is known, we got the following consequence which we could not find any reference for.

Corollary 2.3. *Let $a,q\in\mathbb{C}\setminus\{0\}$, $|q|<1$. Then*

$$\sum_{k=0}^{\infty}\left((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty\right)\frac{q^k}{(q;q)_k} = (a;q)_\infty(q/a;q)_\infty.$$

The following case is commonly called Al-Salam-Carlitz II polynomials, which it is just the Al-Salam-Carlitz polynomials for the $|q|>1$ case.

Theorem 2.4. *Let $a,q\in\mathbb{C}$, $a\neq 0,1$, $|q|>1$. Then, the Al-Salam-Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by*

$$\int_a^1 U_n^{(a)}(x;q^{-1})U_m^{(a)}(x;q^{-1})(q^{-1}x;q^{-1})_\infty(q^{-1}x/a;q^{-1})_\infty d_{q^{-1}}x =$$
$$(-a)^n(1-q^{-1})(q^{-1};q^{-1})_n(q^{-1};q^{-1})_\infty(a;q^{-1})_\infty(q^{-1}/a;q^{-1})_\infty\, q^{-\binom{n}{2}}\delta_{m,n}. \tag{2.3}$$

Proof. Let us denote q^{-1} by p, then $0<|p|<1$. For $a\in\mathbb{C}$, $a\neq 0,1$. Then, by using the identity (2.2) replacing $q\mapsto p$, and taking into account that $w(aq;a;p)=w(q;a;p)=0$ and [4, (14.24.9)], for $m < n$ we get

$$\begin{aligned}\sum_{k=0}^{\infty} & aw(ap^k;a;p)U_m^{(a)}(ap^k;p)U_n^{(a)}(ap^k;p)p^k \\ &= ap^{n-1}\lim_{M\to\infty} U_m^{(a)}(ap^M;p)U_{n-1}^{(a)}(ap^M;p)w(ap^M;a;p) \\ &\quad + ap^{n-1}(1-p^m)\lim_{M\to\infty}\sum_{k=0}^{M-1} aw(ap^k;a;p)U_{n-1}^{(a)}(ap^k;p)U_{m-1}^{(a)}(ap^k;p)p^k.\end{aligned}$$

Following the same idea from the previous result, we get

$$\begin{aligned}\sum_{k=0}^{\infty} & w(p^k;a;p)U_m^{(a)}(p^k;p)U_n^{(a)}(p^k;p)p^k \\ &= ap^{n-1}\lim_{M\to\infty} U_m^{(a)}(p^M;p)U_{n-1}^{(a)}(p^M;p)w(p^M;a;p) \\ &\quad + ap^{n-1}(1-p^m)\lim_{M\to\infty}\sum_{k=0}^{M-1} w(p^k;a;p)U_{n-1}^{(a)}(p^k;p)U_{m-1}^{(a)}(p^k;p)p^k.\end{aligned}$$

Therefore, the property of orthogonality holds for $m < n$. Next, if $n = m$, we have

$$\begin{aligned}\int_a^1 & U_n^{(a)}(x;p)U_n^{(a)}(x;p)w(x;a;p)\,d_px \\ &= \frac{a(p^n-1)}{p^{1-n}}\sum_{k=0}^{\infty}\left(aw(ap^k;a;p)\left(U_{n-1}^{(a)}(ap^k;p)\right)^2 p^k - w(p^k;a;p)\left(U_{n-1}^{(a)}(p^k;p)\right)^2 p^k\right) \\ &= (-a)^n(p;p)_n p^{\binom{n}{2}}\left(\sum_{k=0}^{\infty} aw(ap^k;a;p)p^k - w(p^k;a;p)p^k\right) \\ &= (-a)^n\ (q^{-1};q^{-1})_n(p;p)_\infty p^{\binom{n}{2}}\sum_{k=0}^{\infty}\left(a(p^{k+1}a;p)_\infty - (p^{k+1}/a;p)_\infty\right)\frac{q^k}{(p;p)_k} \\ &= (-a)^n(q^{-1};q^{-1})_n(p;p)_\infty(a;p)_\infty(p/a;p)_\infty p^{\binom{n}{2}}.\end{aligned}$$

Using the same argument as in Theorem 2.1, the uniqueness holds, so the claim follows. □

Remark 2.5. Observe that in the previous theorems if $a = q^m$, with $m \in \mathbb{Z}$, $a \neq 0$, after some logical cancellations, the set of points where we need to calculate the q-integral is easy to compute. For example, if $0 < aq < 1$ and $0 < q < 1$, one obtains the sum [4, p. 537, (14.25.2)].

Remark 2.6. The $a = 1$ case is special because it is not considered in the literature. In fact, the linear form associated to Al-Salam-Carlitz polynomials, namely $\mathbf{u}$, fulfills the Pearson-type distributional equations

$$\mathscr{D}_q[(x-1)^2\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u} \quad \text{and} \quad \mathscr{D}_{q^{-1}}[q^{-1}\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u}.$$

Moreover, the Al-Salam-Carlitz polynomials fulfill the three-term recurrence relation [4, (14.24.3)]

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^nU_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q), \tag{2.4}$$

where $n = 0, 1, \ldots$, with initial conditions $U_0^{(a)}(x;q) = 1$, $U_1^{(a)}(x;q) = x - a - 1$.

Therefore, we believe that it will be interesting to study such a case for its peculiarity because the coefficient $q^{n-1}(1-q^n) \neq 0$ for all n, so one can apply Favard's result.

2.1. **The $|q| = 1$ case.** In this section we only consider the case where q it is a root of unity.

Let N be a positive integer so that $q^N = 1$ then, due to the recurrence relation (2.4) and following the same idea that the authors did in [3, Section 4.2], we apply the following process:

(1) The sequence $(U_n^{(a)}(x;q))_{n=0}^{N-1}$ is orthogonal with respect to the Gaussian quadrature

$$\langle \mathbf{v}, p\rangle := \sum_{s=1}^{N} \gamma_1^{(a)} \dots \gamma_{N-1}^{(a)} \frac{p(x_s)}{\left(U_{N-1}^{(a)}(x_s)\right)^2},$$

where $\{x_1, x_2, \dots, x_N\}$ are the zeros of $U_N^{(a)}(x;q)$ for such value of q.

(2) Since $\langle \mathbf{v}, U_n^{(a)}(x;q) U_n^{(a)}(x;q)\rangle = 0$, we need to modify such a linear form.

Next, we can prove that the sequence $(U_n^{(a)}(x;q))_{n=0}^{2N-1}$ is orthogonal with respect to the bilinear form

$$\langle p, r\rangle_2 = \langle \mathbf{v}, pq\rangle + \langle \mathbf{v}, \mathscr{D}_q^N p \mathscr{D}_q^N r\rangle,$$

since $\mathscr{D}_q U_n^{(a)}(x;q) = (q^n - 1)/(q-1) U_{n-1}^{(a)}(x;q)$.

(3) Since $\langle U_{2N}^{(a)}(x;q), U_{2N}^{(a)}(x;q)\rangle_2 = 0$ and taking into account the what we did before, we consider the linear form

$$\langle p, r\rangle_3 = \langle \mathbf{v}, pq\rangle + \langle \mathbf{v}, \mathscr{D}_q^N p \mathscr{D}_q^N r\rangle + \langle \mathbf{v}, \mathscr{D}_q^{2N} p \mathscr{D}_q^{2N} r\rangle.$$

(4) Therefore one can obtain a sequence of bilinear forms so that the Al-Salam-Carlitz polynomials are orthogonal with respect to them.

3. A Generalized generating function for Al-Salam-Carlitz polynomials

For this section, we are going to assume $|q| > 1$, or $0 < |p| < 1$. Indeed, by starting with the generating functions for Al-Salam-Carlitz polynomials [4, (14.25.11-12)], we derive generalizations using the connection relation for these polynomials.

Theorem 3.1. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$. Then*

$$U_n^{(a)}(x;p) = (-1)^n (p;p)_n p^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a;p)_{n-k} p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k} U_k^{(b)}(x;p). \tag{3.1}$$

Proof. If we consider the generating function for Al-Salam-Carlitz polynomials [4, (14.25.11)]

$$\frac{(xt;p)_\infty}{(t, at;p)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n,$$

and multiply both sides by $(bt;p)_\infty/(bt;p)_\infty$, obtaining

$$\sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n = \frac{(bt;p)_\infty}{(at;p)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(b)}(x;p) t^n. \tag{3.2}$$

If we now apply the q-binomial theorem [4, (1.11.1)]

$$\frac{(az;p)_\infty}{(z;p)_\infty} = \sum_{k=0}^{\infty} \frac{(ap;p)_n}{(p;p)_n} z^n, \quad 0 < |p| < 1, \quad |z| < 1,$$

to (3.2), and then collect powers of t, we obtain

$$\sum_{k=0}^{\infty} t^k \sum_{m=0}^{k} \frac{(-1)^m a^{k-m} (b/a;p)_{k-m} p^{\binom{m}{2}}}{(p;p)_{k-m}(p;p)_m} U_m^{(b)}(x;p) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{n}{2}}}{(p;p)_n} U_n^{(a)}(x;p) t^n.$$

Taking into account this expression, the result follows. ∎

Theorem 3.2. *Let* $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$, $t \in \mathbb{C}$, $|at| < 1$. *Then*

$$(at;p)_\infty\,{}_1\phi_1\left(\begin{matrix} x \\ at \end{matrix};p,t\right) = \sum_{k=0}^{\infty} \frac{p^{k(k-1)}}{(p;p)_k}\,{}_1\phi_1\left(\begin{matrix} b/a \\ 0 \end{matrix};p,atp^k\right) U_k^{(b)}(x;p)t^k. \tag{3.3}$$

Proof. We start with a generating function for Al-Salam-Carlitz polynomials [4, (14.25.12)]

$$(at;q)_\infty\,{}_1\phi_1\left(\begin{matrix} x \\ at \end{matrix};q,t\right) = \sum_{k=0}^{\infty} \frac{q^{n(n-1)}}{(q;q)_n} V_n^{(a)}(x;q)t^n$$

and (3.1) to obtain

$$(at;p)_\infty\,{}_1\phi_1\left(\begin{matrix} x \\ at \end{matrix};p,t\right) = \sum_{n=0}^{\infty} t^n(-1)^n p^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k}(b/a;p)_{n-k}p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k} U_k^{(b)}(x;p).$$

If we reverse the order of summations, shift the n variable by a factor of k, using the basic properties of the q-Pochhammer symbol, and [4, (1.10.1)], we can reverse the order of summation since our sum is of the form

$$\sum_{n=0} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x;p),$$

where

$$a_n = t^n, \qquad c_{n,k} = \frac{(-1)^k a^{n-k}(b/a;p)_{n-k}p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k}.$$

In this case, one has

$$|a_n| \leq |t|^n, \quad |c_{n,k}| \leq K(1+n)^{\sigma_1}|a|^n,$$

and $|U_n^{(a)}(x;p)| \leq (1+n)^{\sigma_2}$, where K_1, σ_1, and σ_2 are positive constants independent on n. Therefore, if $|at| < 1$, then

$$\left|\sum_{n=0} a_n \sum_{k=0}^{n} c_{n,k} U_k^{(a)}(x;p)\right| < \infty,$$

and this completes the proof. □

As we saw in Section 2, the orthogonality relation for Al-Salam-Carlitz polynomials for $|q| > 1$, $|p| < 1$, and $a \neq 0, 1$ is

$$\int_\Gamma U_n^{(a)}(x;p)U_m^{(a)}(x;p)w(x;a;p)d_px = d_n^2\delta_{n,m}.$$

Taking this result in mind, the following result follows.

Theorem 3.3. *Let* $a, b, p \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $|at| < 1$, $|p| < 1$, $m \in \mathbb{N}_0$. *Then*

$$\begin{aligned}\int_a^1 {}_1\phi_1\left(\begin{matrix} q^{-x} \\ at \end{matrix};q,t\right) U_m^{(b)}(q^{-x};p)(q^{-1}x;q^{-1})_\infty(q^{-1}x/a;q^{-1})_\infty dq^{-1} \\ = \big(-bt\big)^m q^{3\binom{m}{2}}(b;p)_\infty(p/b;p)_\infty\,{}_1\phi_1\left(\begin{matrix} b/a \\ 0 \end{matrix};q,atq^m\right).\end{aligned}$$

Proof. From (3.3), we replace $x \mapsto p^x$ and multiply both sides by $U_m^{(b)}(x;p)w(x;a;p)$, and by using the orthogonality relation (2.3), the desired result holds. □

Note that the application of connection relations to the rest of the known generating functions for Al-Salam-Carlitz polynomials [4, (14.24.11), (14.25.11)] leave these generating functions invariant.

References

[1] W. A. Al-Salam and L. Carlitz. Some orthogonal q-polynomials. *Mathematische Nachrichten*, 30:47–61, 1965.

[2] T. S. Chihara. *An introduction to orthogonal polynomials.* Gordon and Breach Science Publishers, New York-London-Paris, 1978. Mathematics and its Applications, Vol. 13.

[3] R. S. Costas-Santos and J. F. Sánchez-Lara. Orthogonality of q-polynomials for non-standard parameters. *Journal of Approximation Theory*, 163(9):1246–1268, 2011.

[4] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder.

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