Maximal Meixner generalized generating functions for Meixner polynomials

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Abstract. We use and derive connection and connection-type relations for Meixner and Krawtchouk polynomials. These relations are used to derive generalizations of generating functions for these orthogonal polynomials. The coefficients of these generalized generating functions are given in term of double hypergeometric functions. From these generalized generating functions, we derive corresponding infinite series expressions by using the orthogonality relations.

Key words: Generating functions; Connection coefficients; Connection-type relations; Eigenfunction expansions; Definite integrals; Infinite series

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1 Introduction

Orthogonal polynomials are a group of polynomial families such that any two different polynomials in that family are orthogonal to each other under some inner product. This relation can sometimes be expressed discretely for a sequence of orthogonal polynomials. For instance, given $\{P_n(x;\mathbf{a})\}$, $n \in \mathbb{N}_0$, with discrete weight $w_x \in \mathbb{C}$, $\mathbf{a}$ is a set of free parameters, and $r_n \in \mathbb{C}$, then one may have the following discrete orthogonality relation

$$\sum_{x=0}^{\infty} P_m(x;\mathbf{a})P_n(x;\mathbf{a})w_x(\mathbf{a}) = r_n(\mathbf{a})\delta_{m,n}.$$

In this paper we discuss connection and connection-type relations, and generalizations of generating functions from these relations for a family of discrete hypergeometric orthogonal polynomials, namely the Meixner and Krawtchouk polynomials [6, Sections 9.10-11].

The paper is organized as follows. In Section 2, some mathematical preliminaries which are used in our proofs are introduced. In Section 3, connection and connection-type relations are given for Meixner and Krawtchouk polynomials. In Section 4, generalizations of generating functions for Meixner and Krawtchouk polynomials are presented. In Section 5, infinite series expressions are given which are derived using orthogonality for Meixner and Krawtchouk polynomials.

2 Preliminaries: hypergeometric functions

Our generalizations of generating functions rely on Pochhammer symbols. The Pochhammer symbol, also called the shifted factorial, is a special function that is used to express coefficients of polynomials. They can be used to express binomial coefficients, coefficients of derivatives of polynomials, and are integral to the definition of hypergeometric functions. The Pochhammer symbol is defined for $a \in \mathbb{C}$, $n \in \mathbb{N}_0$, such that

$$(a)_n := (a)(a+1)\cdots(a+n-1), \tag{2.1}$$

where as have assumed (and throughout this paper) that the empty product is unity. Define

$$\widehat{\mathbb{C}} := \{z \in \mathbb{C} : -z \notin \mathbb{N}_0\},$$
$$\mathbb{C}_0 := \{z \in \mathbb{C} : z \neq 0\},$$
$$\mathbb{C}_{0,1} := \{z \in \mathbb{C} : z \notin \{0,1\}\}.$$

One has the following useful identities for Pochhammer symbols, namely for $n \in \mathbb{N}_0$,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.2}$$

$$\Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(-a+1)_n}, \tag{2.3}$$

where $a \in \widehat{\mathbb{C}}$, and for $k \in \mathbb{N}_0$, $a \in \mathbb{C}$, one has

$$(a)_{n+k} = (a)_n (a+n)_k = (a)_k (a+k)_n. \tag{2.4}$$

Moreover, for many of the proofs in this paper, we will need the following inequalities for Pochhammer symbols [2, Lemma 12]. Let $j \in \mathbb{N}$, $k, n \in \mathbb{N}_0$, $z \in \mathbb{C}$, $\Re u > 0$, $w > -1$, $v \geq 0$. Then

$$|(u)_j| \geq (\Re u)(j-1)!, \tag{2.5}$$

$$\frac{(v)_n}{n!} \leq (1+n)^v, \tag{2.6}$$

$$(n+w)_k \leq \max\{1, 2^w\} \frac{(n+k)!}{n!}, \tag{2.7}$$

$$(z+k)_{n-k} \leq \frac{n!}{k!}(1+n)^{|z|}. \tag{2.8}$$

The generalized generating functions we present in this paper often have coefficients which can be expressed in terms of generalized hypergeometric functions. Generalized hypergeometric functions ${}_rF_s$ are special functions which can be represented by a hypergeometric series. These are solutions of a $\max(s+1,r)^{\mathrm{th}}$ order differential equation with three regular singular points. The generalized hypergeometric function can be defined as [6, (1.4.1)]

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. \tag{2.9}$$

For instance, we often take advantage of the binomial theorem [6, (1.5.1)] which can be expressed as

$$ {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) = (1-z)^{-a}, \qquad |z| < 1. \tag{2.10}$$

Sometimes, the coefficients of our generalized generating funcitons are given in terms of double and triple hypergeometric functions. There exists a large classification of such functions. The versions of these functions which we encounter are given as follows. For double hypergeometric series we encounter the function F_1 which is an Appell series. These are hypergeometric series in two variables and are defined as [4, (16.13.1)]

$$F_1\left(a, b, b'; c; x, y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}. \tag{2.11}$$

We also encounter the function Φ_2, which is a Humbert hypergeometric series of two variables defined as [7, p. 25]

$$\Phi_2\left(\beta, \beta'; \gamma; x, y\right) := \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}. \tag{2.12}$$

The function $F_D^{(3)}$, a hypergeometric function of three-variables, is a form of the triple Lauricella series defined as [7, p. 33]

$$F_D^{(3)}\left(a, b_1, b_2, b_3; c; x, y, z\right) := \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{2.13}$$

The function $\Phi_2^{(3)}$ is a confluent form of the triple Lauricella series defined as [7, p. 34]

$$\Phi_2^{(3)}\left(b_1, b_2, b_3; c; x, y, z\right) := \sum_{m,n,p=0}^{\infty} \frac{(b_1)_m\,(b_2)_n\,(b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{2.14}$$

3 Connection and connection-type relations

The Meixner polynomials are defined as [6, (9.10.1)]

$$M_n(x; \alpha, c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \alpha \end{matrix}; 1 - \frac{1}{c}\right). \tag{3.1}$$

In this section we derive connection and connection-type (see Remark 2) relations for Meixner polynomials. For the entire paper, we assume that $x \in \mathbb{C}$, $n \in \mathbb{N}_0$. The following connection relations for Meixner polynomials can be found in Gasper (1974) [5, (5.2-5)].

Theorem 1. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$. *Then*

$$M_n(x; \alpha, c) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta)_k}{(\alpha)_k} \left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix} -n+k, k+\beta \\ k+\alpha \end{matrix}; \frac{d(1-c)}{c(1-d)}\right) M_k(x; \beta, d). \tag{3.2}$$

By setting $\beta = \alpha$ in (3.2) one obtains the following corollary.

Corollary 2. *Let* $\alpha \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$. *Then*

$$M_n(x; \alpha, c) = \left(\frac{c-d}{c(1-d)}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d(1-c)}{c-d}\right)^k M_k(x; \alpha, d). \tag{3.3}$$

Furthermore by setting $d = c$ in (3.2) and using the Gauss formula [4, (15.4.20)], one obtains the following corollary.

Corollary 3. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c \in \mathbb{C}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}(\alpha-\beta)_{n-k}(\beta)_k M_k(x;\beta,c). \tag{3.4}$$

Remark 1. Note that even though Theorem 1 was originally stated in [5] for $\alpha > 0$, $c \in (0,1)$, one can extend [5, (5.9-12)] analytically for $\alpha, \beta \in \mathbb{C}$, $-\alpha, -\beta \notin \mathbb{N}_0$, $c, d \in \mathbb{C}_{0,1}$, since, in such a case, one loses normality of the polynomials, i.e., $\deg M_n(x) < n$ for some n. However, formally, Theorem 1 remains true for $c = 1$, and all β, d in the above domains.

Remark 2. By *connection-type* relations for orthogonal polynomials, we mean a relation where the left-hand side is an orthogonal polynomial with argument x and set of parameters $\mathbf{a}$, and the right-hand side is given by a finite sum over coefficients which in general may depend on x, multiplied by a product of that same polynomial with a set of different parameters $\mathbf{b}$, namely

$$P_n(x;\mathbf{a}) = \sum_{k=0}^{n}\alpha_{k,n}(x;\mathbf{a},\mathbf{b})P_k(x;\mathbf{b}).$$

Connection-type relations are not connection relations because the coefficients multiplying the orthogonal polynomials depend on the argument. For connection relations, the coefficients of the orthogonal polynomials must not depend on the argument.

We now derive a connection-type relation for Meixner polynomials corresponding to the parameter c.

Theorem 4. *Let* $\alpha \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k(x)_{n-k}}{d^{n-k}}\,{}_2F_1\left(\begin{matrix}-n+k,-x\\-x+k-n+1\end{matrix};\frac{d}{c}\right)M_k(x;\alpha,d). \tag{3.5}$$

Proof. A generating function for Meixner polynomials is given as [6, (9.10.11)]

$$\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n, \qquad |t|<|c|<1. \tag{3.6}$$

The above connection-type relation (3.5) can be derived by starting with (3.6), and multiplying the left-hand side by $\left(1-\frac{t}{d}\right)^x \Big/ \left(1-\frac{t}{d}\right)^x$, $|t|<|d|<1$. One then has

$$\left(1-\frac{t}{c}\right)^x\left(1-\frac{t}{d}\right)^{-x}\left(1-\frac{t}{d}\right)^x(1-t)^{-x-\alpha} = \left(1-\frac{t}{c}\right)^x\left(1-\frac{t}{d}\right)^{-x}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}M_m(x;\alpha,d)t^m.$$

After using the binomial theorem (2.10), the left-hand side becomes

$$\sum_{k=0}^{\infty}\frac{(-x)_k}{k!}\left(\frac{t}{c}\right)^k\sum_{s=0}^{\infty}\frac{(x)_s}{s!}\left(\frac{t}{d}\right)^s\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}M_m(x;\alpha,d)t^m.$$

By collecting the terms associated with t^n, (3.5) follows using analytic contination in c, d, and (2.2), (2.3) and (2.9). ∎

We now derive a connection-type relation for Meixner polynomials corresponding to free parameters α, c. The theorem below is not a connection relation because the coefficients multiplied by the Meixner polynomials depend on x (see Remark 2).

Theorem 5. *Let $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$. Then*

$$M_n(x;\alpha,c) = \frac{(\alpha-\beta)_n}{(\alpha)_n}\sum_{k=0}^{n}\frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k}F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right) \times M_k(x;\beta,d), \tag{3.7}$$

where F_1 is given by (2.11).

Proof. We substitute the connection relation for the free parameter α (3.4) with the connection-type relation for the free parameter d (3.5) to obtain the result

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}(\alpha-\beta)_{n-k}(\beta)_k\frac{k!}{(\beta)_k}\sum_{m=0}^{k}\frac{(\beta)_m(x)_{k-m}}{m!(k-m)!d^{m-k}} \times {}_2F_1\left(\begin{matrix}-k+m,-x\\ -x+m-k+1\end{matrix};\frac{d}{c}\right)M_m(x;\beta,d).$$

If we expand the hypergeometric, switch the order of summations twice, and use (2.2)-(2.3), (2.9), and (2.11) the result follows. ∎

We have just found a finite expansion of the Meixner polynomials with free parameters α, c in terms of Meixner polynomials with free parameters β, d. The coefficient of the connection-type relation depends on x, so this is not a connection relation. We now combine generating functions for Meixner polynomials with that connection-type relation to derive generalized generating functions.

In the other hand, the Krawtchouk polynomials are a particular case of Meixner polynomials. In fact, they are related in the following way:

$$K_n(x;p,N) = M_n\left(x;-N,\frac{p}{p-1}\right). \tag{3.8}$$

Taking this into account, we can write them as a truncated hypergeometric series as [6, (9.11.1)]

$$K_n(x;p,N) := {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right). \tag{3.9}$$

The following results can be found in [5, (5.9-10), (5.11-12)].

Theorem 6. *Let $n, M, N \in \mathbb{N}_0$, $n \le N \le M$, $p, q \in \mathbb{C}_0$. Then*

$$K_n(x;p,N) = \sum_{k=0}^{n}\binom{n}{k}\frac{q^k(-M)_k}{p^k(-N)_k}\,{}_2F_1\left(\begin{matrix}-n+k,k-M\\ k-N\end{matrix};\frac{q}{p}\right)K_k(x;q,M). \tag{3.10}$$

Corollary 7. *Let $n, N \in \mathbb{N}_0$, $n \le N$, $p, q \in \mathbb{C}_0$. Then*

$$K_n(x;p,N) = \left(\frac{p-q}{p}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{q}{p-q}\right)^k K_k(x;q,N). \tag{3.11}$$

Furthermore by setting $d = c$ in (3.2) and using the Gauss formula [4, (15.4.20)], one obtains

Corollary 8. *Let* $n, M, N \in \mathbb{N}_0$, $n \le N \le M$, $p, q \in \mathbb{C}_0$. *Then*

$$K_n(x;p,N) = \frac{1}{(-N)_n} \sum_{k=0}^{n} \binom{n}{k} (M-N)_{n-k}(-M)_k K_k(x;p,M). \tag{3.12}$$

Remark 3. Observe that the previous results can be obtained from (3.1) by setting the right values and using the relation (3.8).

We now combine generating functions for Krawtchouk polynomials with these connection relations to derive generalized generating functions.

4 Generalized generating functions

We now derive generalized generating functions for the Meixner polynomials.

Theorem 9. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$, $t \in \mathbb{C}$. *Then*

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!} \left(\frac{d(1-c)}{c(1-d)}\right)^n {}_1F_1\left(\begin{matrix} \beta+n \\ \alpha+n \end{matrix}; \frac{-td(1-c)}{c(1-d)}\right) M_n(x;\beta,d) t^n. \tag{4.1}$$

Proof. Using the generating function for Meixner polynomials [6, (9.10.12)]

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} M_k(x;\beta,d) \frac{t^n}{n!}$$

and (3.2), we obtain

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right)$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta)_k}{(\alpha)_k} \left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix} -n+k, \beta+k \\ \alpha+k \end{matrix}; \frac{d(1-c)}{c(1-d)}\right) M_k(x;\beta,d).$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, switch the order of summations again, and use (2.2)-(2.3) and (2.9). Again, in order to justify reversing the summation symbols it is enough to show that

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,d) \right| < \infty,$$

where $|M_k(x,\beta,d)| \le K_1(1+k)^{\sigma_2} d^{-k}$, $a_n = t^n/n!$, hence $|a_n| \le |t|^n/n!$, and

$$c_{k,n} = \sum_{s=0}^{n-k} \frac{(-1)^k (-n)_{s+k} (\beta)_{s+k}}{(\alpha)_{s+k} k! s!} \left(\frac{d(1-c)}{c(1-d)}\right)^{s+k},$$

where K_1 and σ_1 are positive constants not depending on n. Then since

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,d) \right| \le K_1 K_2 \sum_{n=0}^{\infty} \frac{(1+n)^{\sigma_1+\sigma_2+1}}{n!} \left|\frac{t}{c}\right|^n \left|\frac{1+d-2c}{1-d}\right|^n < \infty,$$

the result follows because all the sums connected with these coefficients converge. ■

The next result is a direct consequence of Theorem 9.

Theorem 10. *Let $\alpha, \beta \in \widehat{\mathbb{C}}$, $c \in \mathbb{C}_{0,1}$, $t \in \mathbb{C}$. Then*

$$e^t {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!}\, {}_1F_1\left(\begin{matrix}\alpha-\beta\\ \alpha+n\end{matrix};t\right) M_n(x;\beta,c)t^n. \tag{4.2}$$

We have just found a finite expansion of the Meixner polynomials with free parameter c in terms of Meixner polynomials with free parameter d. The coefficient of the connection-type relation depends on x, so this is not a connection relation. We now combine Meixner generating function [6, (9.10.12)] with that connection-type relation to derive a generalized generating function.

Theorem 11. *Let $\alpha \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$, $t \in \mathbb{C}$. Then*

$$e^t {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi_2\left(x,-x;\alpha+n;\frac{t}{c},\frac{t}{d}\right) M_n(x;\alpha,d)t^n, \tag{4.3}$$

where Φ_2 is given by (2.12).

Proof. Using [6, (9.10.12)] and (3.5), we obtain

$$\begin{aligned} e^t {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right) &= \sum_{n=0}^{\infty} \frac{t^n}{(\alpha)_n} \sum_{k=0}^{n} \frac{(\alpha)_k (x)_{n-k}}{k!(n-k)!d^{n-k}} M_k(x;\alpha,d) \\ &\quad \times {}_2F_1\left(\begin{matrix}-n+k,-x\\ -x+k-n+1\end{matrix};\frac{d}{c}\right). \end{aligned} \tag{4.4}$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the hypergeometric, and use (2.2)-(2.3), (2.9), and (2.12). We can justify the reversing the summation symbols since in this case

$$a_n = \frac{t^n}{n!}, \quad \text{and} \quad c_{k,n} = \binom{n}{k}\frac{(\alpha)_k (x)_{n-k}}{d^{n-k}}\, {}_2F_1\left(\begin{matrix}-n+k,-x\\ -x+k-n+1\end{matrix};\frac{d}{c}\right).$$

Therefore

$$\sum_{n=0}^{\infty} |a_n| \left|\sum_{k=0}^{n} c_{k,n} M_k(x;\alpha,d)\right| \le K_3 \sum_{n=0}^{\infty} \frac{(1+n)^{\sigma_3}}{n!}\left|\frac{t}{c}\right|^n,$$

where K_3 and σ_3 are positive constants not depending on n, then the result holds since all these sums connected with these coefficients converge. ∎

Theorem 12. *Let $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \mathbb{C}_{0,1}$, $t \in \mathbb{C}$. Then*

$$e^t {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!}\Phi_2^{(3)}\left(x,-x,\alpha-\beta;\alpha+n;\frac{t}{c},\frac{t}{d},t\right) M_n(x;\beta,d)t^n, \tag{4.5}$$

where $\Phi_2^{(3)}$ is given in (2.14).

Proof. Using [6, (9.10.12)] and (3.7), we obtain

$$\begin{aligned} e^t {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{(\alpha-\beta)_n}{(\alpha)_n} \sum_{k=0}^{n} \frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k} M_k(x;\beta,d) \\ &\quad \times F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right). \end{aligned} \tag{4.6}$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k,expand the Appell series, switch the order of summations two more times, and use (2.2)-(2.3), (2.9), and (2.14). Indeed,

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,d) \right| \le K_4 \sum_{n=0}^{\infty} \frac{(1+n)^{\sigma_4}}{n!} \left| \frac{t(c+d)}{cd} \right|^n < \infty,$$

where K_4 and σ_4 are positive constants not depending on n, then the result holds since all these sums connected with these coefficients can be rearranged in the desired way. ■

We also have the connection relation with one free parameter given by (3.4). We now combine this connection relation with the above referenced generating functions to obtain new generalized generating functions for Meixner polynomials.

Theorem 13. *Let* $c \in \mathbb{C}_{0,1}$, $\gamma, t \in \mathbb{C}$, $|t| < 1$, $|t(1-c)| < |c(1-t)|$, $\alpha, \beta \in \widehat{\mathbb{C}}$. *Then*

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix} \gamma, -x \\ \alpha \end{matrix} ; \frac{t(1-c)}{c(1-t)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (\beta)_n}{(\alpha)_n n!} \, {}_2F_1\left(\begin{matrix} \gamma+n, \alpha-\beta \\ \alpha+n \end{matrix} ; t \right) M_n(x;\beta,c) t^n. \quad (4.7)$$

Proof. Using the generating function for Meixner polynomials [6, (9.10.13)] and (3.4), we obtain

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix} \gamma, -x \\ \alpha \end{matrix} ; \frac{t(1-c)}{c(1-t)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n t^n}{(\alpha)_n} \sum_{k=0}^{n} \frac{(\alpha-\beta)_{n-k} (\beta)_k}{(n-k)! k!} M_k(x;\beta,c).$$

If we switch the order of summations, shift the n variable by a factor of k and use (2.2)-(2.3) and (2.9). Indeed, in this case $a_n = t^n (\gamma)_n / n!$, therefore

$$|a_n| \le |t|^n (1+n)^{|\gamma|}.$$

So, we have

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,c) \right| \le K_5 \sum_{n=0}^{\infty} (1+n)^{\sigma_5} \left| \frac{t(1-c)}{c(1-t)} \right|^n,$$

where K_5 and σ_5 are positive constants not depending on n. Therefore if $|t| < 1$ and $|t(1-c)| < |c(1-t)|$ the sum converges, then the result holds since all these sums connected with these coefficients can be rearranged in the desired way. ■

Theorem 14. *Let* $c, d \in \mathbb{C}_{0,1}$, $\gamma, t \in \mathbb{C}$, $|t| < \min\{1, |c(1-d)|/|1+d-2c|\}$, $\alpha, \beta \in \widehat{\mathbb{C}}$. *Then*

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix} \gamma, -x \\ \alpha \end{matrix} ; \frac{t(1-c)}{c(1-t)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (\beta)_n}{(\alpha)_n n!} \, {}_2F_1\left(\begin{matrix} \gamma+n, \beta+n \\ \alpha+n \end{matrix} ; \frac{-dt(1-c)}{c(1-d)(1-t)} \right) \times \left(\frac{d(1-c)}{c(1-d)(1-t)} \right)^n M_n(x;\beta,d) t^n. \quad (4.8)$$

Proof. Using [6, (9.10.13)] and (3.2), we obtain

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix} \gamma, -x \\ \alpha \end{matrix} ; \frac{t(1-c)}{c(1-t)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} t^n \sum_{k=0}^{n} \frac{(\beta)_k n!}{k!(n-k)!(\alpha)_k} \left(\frac{d(1-c)}{c(1-d)} \right)^k M_k(x;\beta,d) \times {}_2F_1\left(\begin{matrix} -n+k, \beta+k \\ \alpha+k \end{matrix} ; \frac{d(1-c)}{c(1-d)} \right).$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, switch the order of summations again, and use (2.2)-(2.3) and (2.9), then the result holds since all these sums connected with these coefficients converge (it is similar to the previous proof combined with the proof of Theorem 9) and can be rearranged in the desired way. ■

We have just found a finite expansion of the Meixner polynomials with free parameter c in terms of Meixner polynomials with free parameter d. The coefficient of the connection-type relation depends on t, so this is not a connection relation. We now combine Meixner generating function [6, (9.10.13)] with that connection-type relation to derive a generalized generating function.

Theorem 15. *Let* $|t| < \min\{1, |c|\}, \alpha \in \widehat{\mathbb{C}}$, $\gamma \in \mathbb{C}$, $c, d \in \mathbb{C}_{0,1}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} F_1\left(\gamma+n, x, -x; \alpha+n; \frac{t}{c}, \frac{t}{d}\right) M_n(x;\alpha,d)t^n. \quad (4.9)$$

Proof. Using [6, (9.10.13)] and (3.5), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n t^n}{(\alpha)_n n!} \sum_{k=0}^{n} \frac{(\alpha)_k (x)_{n-k}}{k!(n-k)! d^{n-k}} M_k(x;\alpha,d)$$
$$\times {}_2F_1\left(\begin{matrix}-n+k, -x\\ -x+k-n+1\end{matrix}; \frac{d}{c}\right).$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the hypergeometric, and use (2.2)-(2.3), (2.9), and (2.11), then the result holds since all these sums connected with these coefficients converge (it is similar to the proof of Theorem 13 combined with the proof of Theorem 11) and can be rearranged in the desired way. ■

Theorem 16. *Let* $|t| < \min\{1, |cd|/|c+d|\}, \alpha, \beta \in \widehat{\mathbb{C}}$, $\gamma \in \mathbb{C}$, $c, d \in \mathbb{C}_{0,1}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n(\gamma)_n}{(\alpha)_n n!} F_D^{(3)}\left(\gamma+n, x, -x, \alpha-\beta; \alpha+n; \frac{t}{c}, \frac{t}{d}, t\right) \times M_n(x;\beta,d)t^n, \quad (4.10)$$

where $F_D^{(3)}$ *is given in (2.13).*

Proof. Using [6, (9.10.13)] and (3.7), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n t^n}{n!} \frac{(\alpha-\beta)_n}{(\alpha)_n} \sum_{k=0}^{\infty} \frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k} M_k(x;\beta,d)$$
$$\times F_1\left(-n+k, -x, x; \beta-\alpha-n+k+1; \frac{1}{c}, \frac{1}{d}\right). \quad (4.11)$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k,expand the Appell series, switch the order of summations two more times, and use (2.2)-(2.3), (2.9), and (2.13),then the result holds since all these sums connected with these coefficients converge (it is similar to the proof of Theorem 13 combined with the proof of Theorem 17) and can be rearranged in the desired way. ■

We have derived generalized generating functions for the free parameter c. However, since the coefficients of our connection-type relation is in terms of x, we cannot use the orthogonality relation to create new infinite sums. Note that the application of connection relations (3.4) and (3.5) to the rest of the known generating functions for Meixner polynomials [6, (9.10.11-13)] leave these generating functions invariant. We now derive generalized generating functions for the Krawtchouk polynomials, where we will need a special notation for some of the generating functions. Let $f \in C^\infty(\mathbb{C})$, $N \in \mathbb{N}_0$, $t \in \mathbb{C}$. Define

$$[f(z)]_N := \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} t^k.$$

Theorem 17. *Let $p \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$. Then*

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix}; -\frac{t}{p}\right)\right]_N = \sum_{n=0}^{M} \frac{(-M)_n}{(-N)_n n!} {}_1F_1\left(\begin{matrix}n-M\\n-N\end{matrix}; -t\right) K_n(x;p,M) t^n. \tag{4.12}$$

Proof. Using the generating function for Krawtchouk polynomials [6, (9.11.12)] and (3.12), we obtain

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix}; -\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{t^n}{n!(-N)_n} \sum_{k=0}^{n} \binom{n}{k} (M-N)_{n-k} (-M)_k K_k(x;p,M) \tag{4.13}$$

If we switch the order of summations, shift the n variable by a factor of k and use (2.2)-(2.3) and (2.9), the proof follows since all the series have finite number of terms. ■

Theorem 18. *Let $p, q \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$. Then*

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix}; -\frac{t}{p}\right)\right]_N = \sum_{n=0}^{M} \frac{(-M)_n}{(-N)_n n!} \left(\frac{q}{p}\right)^n {}_1F_1\left(\begin{matrix}n-M\\n-N\end{matrix}; \frac{-tq}{p}\right) K_n(x;q,M) t^n. \tag{4.14}$$

Proof. Using [6, (9.11.12)] and (3.11), we obtain

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix}; -\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-M)_k q^k}{(-N)_k p^k} {}_2F_1\left(\begin{matrix}-n+k, k-M\\k-N\end{matrix}; \frac{q}{p}\right) K_k(x;q,M). \tag{4.15}$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, then switch the order of summations again and shift the n variable again, and use (2.2)-(2.3) and (2.9), the proof follows since all the series have finite number of terms. ■

Theorem 19. *Let $p \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t, \gamma \in \mathbb{C}$. Then*

$$\left[(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma, -x\\-N\end{matrix}; \frac{t}{p(t-1)}\right)\right]_N = \sum_{n=0}^{M} \frac{(-M)_n (\gamma)_n}{(-N)_n n! (1-t)^n} {}_2F_1\left(\begin{matrix}n-M, \gamma+n\\n-N\end{matrix}; \frac{-t}{1-t}\right) \times K_n(x;p,M) t^n. \tag{4.16}$$

Proof. Using [6, (9.11.12)] and (3.12), we obtain

$$(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma, -x\\-N\end{matrix}; \frac{t}{p(t-1)}\right) = \sum_{n=0}^{N} \frac{(\gamma)_n}{n!(-N)_n} \sum_{k=0}^{n} \binom{n}{k} (M-N)_{n-k} (-M)_k K_k(x;p,M) t^n. \tag{4.17}$$

If we switch the order of summations, shift the n variable by a factor of k and use (2.2)-(2.3) and (2.9), the proof follows since all the series have finite number of terms. ■

Theorem 20. *Let $p, q \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t, \gamma \in \mathbb{C}$. Then*

$$\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ -N\end{matrix}; \frac{t}{p(t-1)}\right)\right]_N = \sum_{n=0}^{M} \frac{(-M)_n(\gamma)_n}{(-N)_n n!}\,{}_1F_1\left(\begin{matrix}n-M\\ n-N, \gamma+n\end{matrix}; \frac{-tq}{p(1-t)}\right) \times \frac{q^n}{(1-t)^n p^n} K_n(x; q, M) t^n. \tag{4.18}$$

Proof. Using [6, (9.11.12)] and (3.11), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x\\ -N\end{matrix}; \frac{t}{p(t-1)}\right) = \sum_{n=0}^{N} \frac{(\gamma)_n}{n!} t^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-M)_k q^k}{(-N)_k p^k}\,{}_2F_1\left(\begin{matrix}-n+k, k-M\\ k-N\end{matrix}; \frac{q}{p}\right) \times K_k(x; q, M). \tag{4.19}$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, then switch the order of summations again and shift the n variable again, and use (2.2)-(2.3) and (2.9), the proof follows since all the series have finite number of terms. ■

Note that the application of connection relations (3.11) and (3.12) to the rest of the known generating functions for Krawtchouk polynomials [6, (9.11.11-13)] leave these generating functions invariant.

5 Results using orthogonality

We have derived generalized generating functions for the free parameter α. We now combine this with the orthogonality relation for Meixner polynomials to produce new results from our generalized generating functions. The well-known orthogonality relation for Meixner polynomials for $n, m \in \mathbb{N}_0$, $\alpha > 0$, $c \in (0, 1)$ is [6, (14.25.2)]

$$\sum_{x=0}^{\infty} M_n(x; \alpha, c) M_m(x; \alpha, c) \frac{\Gamma(x+\alpha) c^x}{\Gamma(x+1)} = \kappa_n \delta_{m,n}, \tag{5.1}$$

where

$$\kappa_n = \frac{n!}{c^n (1-c)^\alpha (\alpha)_n}.$$

Note that this a particular case of a more general property of orthogonality fulfilled by Meixner polynomials (see [3, Proposition 9]).

Proposition 21. *Let $m, n \in \mathbb{N}_0$, $\alpha \in \widehat{\mathbb{C}}$, $c \in \mathbb{C} \setminus [0, \infty)$. The orthogonality relation for Meixner polynomials can be given as*

$$\int_C M_n(z; \alpha, c) M_m(x; \alpha, c) \Gamma(-z) \Gamma(z+\alpha) (-c)^z dz = \kappa_n \delta_{m,n}, \tag{5.2}$$

where C is a complex contour from $-\infty i$ to ∞i separating the increasing poles at $z \in \mathbb{N}_0$ from the decreasing poles at $z \in \{-\alpha, -\alpha-1, -\alpha-2, \dots\}$.

In fact, observe that the case $c > 0$ cannot be considered by an integral of the form (5.2) since it diverges. However, when $|c| < 1$, (5.2) is rewritten on the form (see [8, Section 5.6] for details) presented in (5.1). With this result in mind, the following result and corresponding consequences hold.

Theorem 22. *Let $t\in\mathbb{C}$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $c\in\mathbb{C}\setminus[0,\infty)$. Then*

$$\int_C {}_1F_1\left({-x \atop \alpha};\frac{t(1-c)}{c}\right) M_n(x;\beta,c)\Gamma(-z)\Gamma(z+\alpha)(-c)^z\,dz=\frac{t^n\,e^{-t}}{(1-c)^\beta(\alpha)_nc^n}\,{}_1F_1\left({\alpha-\beta \atop \alpha+n};t\right). \tag{5.3}$$

Proof. From (4.2) we multiply both sides by $M_m(x;\beta,c)w(x;\beta,c)$, where $w(x;\beta,c):=\Gamma(-z)\Gamma(z+\alpha)(-c)^z$, utilizing the orthogonality relation (5.2), produces the desired result. ∎

Corollary 23. *Let $t\in\mathbb{C}$, $\alpha,\beta>0$, $c\in(0,1)$. Then*

$$\sum_{x=0}^\infty {}_1F_1\left({-x \atop \alpha};\frac{t(1-c)}{c}\right) M_n(x;\beta,c)\frac{(\beta)_xc^x}{x!}=\frac{t^n\,e^{-t}}{(1-c)^\beta(\alpha)_nc^n}\,{}_1F_1\left({\alpha-\beta \atop \alpha+n};t\right). \tag{5.4}$$

Corollary 24. *Let $t\in\mathbb{C}$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}\setminus[0,\infty)$. Then*

$$\begin{aligned}\int_C {}_1F_1\left({-x \atop \alpha};\frac{t(1-c)}{c}\right) M_n(x;\beta,d)\Gamma(-z)\Gamma(z+\beta)(-d)^z\,dz&=\frac{t^n(1-c)^ne^{-t}}{(1-d)^{n+\beta}(\alpha)_nc^n}\\&\times{}_1F_1\left({\beta+n \atop \alpha+n};\frac{-dt(1-c)}{c(1-d)}\right).\end{aligned} \tag{5.5}$$

Proof. From (4.1) we multiply both sides by $M_m(x;\beta,c)w(x;\beta,c)$, utilizing the orthogonality relation (5.2). ∎

Corollary 25. *Let $t\in\mathbb{C}$, $\alpha,\beta>0$, $c,d\in(0,1)$. Then*

$$\sum_{k=0}^\infty {}_1F_1\left({-x \atop \alpha};\frac{t(1-c)}{c}\right) M_n(x;\beta,d)\frac{d^x(\beta)_x}{x!}=\frac{t^n(1-c)^ne^{-t}}{c^n(1-d)^{n+\beta}(\alpha)_n}\,{}_1F_1\left({\beta+n \atop \alpha+n};\frac{-dt(1-c)}{c(1-d)}\right). \tag{5.6}$$

Corollary 26. *Let $c\in\mathbb{C}\setminus[0,\infty)$, $t\in\mathbb{C}$, $|t|<1$, $|t(1-c)|<|c(1-t)|$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $\gamma\in\mathbb{C}$. Then*

$$\int_C {}_2F_1\left({\gamma,-x \atop \alpha};\frac{t(1-c)}{c(1-t)}\right) M_n(x;\beta,c)\Gamma(z+\beta)(-c)^z\,dz=\frac{(1-t)^\gamma(\gamma)_nt^n}{(1-c)^\beta(\alpha)_nc^n}\,{}_2F_1\left({\alpha-\beta,\gamma+n \atop \alpha+n};t\right). \tag{5.7}$$

Proof. From (4.7) we multiply both sides by $M_m(x;\beta,c)w(x;\beta,c)$, utilizing the orthogonality relation (5.2). ∎

Corollary 27. *Let $c\in(0,1)$, $t\in\mathbb{C}$, $|t|<1$, $|t(1-c)|<|c(1-t)|$, $\alpha,\beta>0$, $\gamma\in\mathbb{C}$. Then*

$$\sum_{x=0}^\infty {}_2F_1\left({\gamma,-x \atop \alpha};\frac{t(1-c)}{c(1-t)}\right) M_n(x;\beta,c)\frac{c^x(\beta)_x}{x!}=\frac{(1-t)^\gamma(\gamma)_nt^n}{(1-c)^\beta(\alpha)_nc^n}\,{}_2F_1\left({\alpha-\beta,\gamma+n \atop \alpha+n};t\right). \tag{5.8}$$

Corollary 28. *Let $t\in\mathbb{C}$, $|t|<\min\{1,|c(1-d)|/|1+d-2c|\}$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $\gamma\in\mathbb{C}$, $c,d\in\mathbb{C}\setminus[0,\infty)$. Then*

$$\begin{aligned}\int_C {}_1F_1\left({\gamma,-x \atop \alpha};\frac{t(1-c)}{c(1-t)}\right) M_n(x;\beta,d)\Gamma(z+\beta)(-d)^z\,dz&=\frac{(\gamma)_n}{(1-d)^{n+\beta}(\alpha)_n}\left(\frac{t(1-c)}{c(1-t)}\right)^n\\&\times{}_2F_1\left({\gamma+n,\beta+n \atop \alpha+n};\frac{-dt(1-c)}{c(1-d)(1-t)}\right).\end{aligned} \tag{5.9}$$

Proof. From (4.8) we multiply both sides by $M_m(x;\beta,c)w(x;\beta,c)$, utilizing the orthogonality relation (5.1), produces the desired result. ∎

Corollary 29. *Let $c, d \in (0,1)$, $t \in \mathbb{C}$, $|t| < \min\{1, |cd|/|c+d|\}, \alpha, \beta > 0$, $\gamma \in \mathbb{C}$. Then*

$$\sum_{k=0}^{\infty} {}_1F_1\left(\begin{matrix}\gamma, -x \\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) M_n(x;\beta,d)\frac{d^x(\beta)_x}{x!} = \frac{(\gamma)_n}{(1-d)^{n+\beta}(\alpha)_n}\left(\frac{t(1-c)}{c(1-t)}\right)^n \times {}_2F_1\left(\begin{matrix}\gamma+n, \beta+n \\ \alpha+n\end{matrix}; \frac{-dt(1-c)}{c(1-d)(1-t)}\right). \tag{5.10}$$

On the other hand, since the Krawtchouk polynomials satisfy the property of orthogonality

$$\sum_{x=0}^{N} \binom{N}{x} p^x(1-p)^{N-x} K_n(x;p,N)K_m(x;p,N) = \frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n \delta_{n,m},$$

the following identities follow with proofs given as above, which we omit.

Corollary 30. *Let $p \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$. Then*

$$\sum_{x=0}^{M} \binom{M}{x} p^x(1-p)^{M-x}\left[e^t {}_1F_1\left(\begin{matrix}-x \\ -N\end{matrix}; -\frac{t}{p}\right)\right]_N K_n(x;p,M) = \frac{(p-1)^n t^n}{p^n(-N)_n} {}_1F_1\left(\begin{matrix}n-M \\ n-N\end{matrix}; -t\right).$$

Corollary 31. *Let $p, q \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$. Then*

$$\sum_{x=0}^{M} \binom{M}{x} q^x(1-q)^{M-x}\left[e^t {}_1F_1\left(\begin{matrix}-x \\ -N\end{matrix}; -\frac{t}{p}\right)\right]_N K_n(x;q,M) = \frac{(q-1)^n t^n}{p^n(-N)_n} {}_1F_1\left(\begin{matrix}n-M \\ n-N\end{matrix}; \frac{-tq}{p}\right).$$

Corollary 32. *Let $\gamma \in \mathbb{C}$, $p \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$, $|t| < 1$. Then*

$$\sum_{x=0}^{M} \binom{M}{x} p^x(1-p)^{M-x}\left[(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma, -x \\ -N\end{matrix}; \frac{t}{p(t-1)}\right)\right]_N K_n(x;p,M) = \frac{(\gamma)_n(1-p)^n t^n}{(-N)_n(t-1)^n p^n} {}_2F_1\left(\begin{matrix}n-M, \gamma+n \\ n-N\end{matrix}; \frac{-t}{1-t}\right).$$

Corollary 33. *Let $\gamma \in \mathbb{C}$, $p, q, \in \mathbb{C}_0$, $M, N \in \mathbb{N}_0$, $N \le M$, $t \in \mathbb{C}$, $|t| < 1$. Then*

$$\sum_{x=0}^{M} \binom{M}{x} q^x(1-q)^{M-x}\left[(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma, -x \\ -N\end{matrix}; \frac{t}{p(t-1)}\right)\right]_N K_n(x;q,M) = \frac{(\gamma)_n(q-1)^n t^n}{(-N)_n p^n(1-t)^n} {}_1F_1\left(\begin{matrix}n-M \\ n-N, \gamma+n\end{matrix}; \frac{-tq}{p(1-t)}\right).$$

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