Analytic properties of some basic hypergeometric-Sobolev-type orthogonal polynomials

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Abstract. In this contribution we consider sequences of monic polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle f, g\rangle_S := \langle \mathbf{u}, fg\rangle + N(\mathscr{D}_q f)(\alpha)(\mathscr{D}_q g)(\alpha), \qquad \alpha \in \mathbb{R}, \quad N \geq 0,$$

where $\mathbf{u}$ is a q-classical linear functional and $\mathscr{D}_q$ is the q-derivative operator.
We obtain some algebraic properties of these polynomials such as an explicit representation, a five-term recurrence relation as well as a second order linear q-difference holonomic equation fulfilled by such polynomials.
We present an analysis of the behaviour of its zeros as a function of the mass N. In particular, we obtain the exact values of N such that the smallest (respectively, the greatest) zero of the studied polynomials is located outside of the support of the measure.
We conclude this work considering two examples.

Key words: Classical orthogonal polynomials; Sobolev-type orthogonal polynomials; basic Hypergeometric series; zeros.

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1 Introduction

The study of polynomial sequences orthogonal with respect to an inner product involving differences was started in two papers [?, ?]. H. Bavinck considered the inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)d\mu(t) + \lambda(\Delta p)(c)(\Delta q)(c), \tag{1}$$

where p, q are polynomials with real coefficients, $c \in \mathbb{R}$, μ is a distribution function with infinite support such that μ has no points of increase in the interval $(c, c+1)$, $\lambda \in \mathbb{R}_+$, and where $(\Delta p)(c) = p(c+1) - p(c)$ denotes the forward difference operator.

Later on, in [?] the authors obtained a difference operator of infinite order for which these orthogonal polynomials (called Sobolev-type Meixner polynomials) are eigenfunctions. The name Sobolev-type is justified from the analogy with the case

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)d\mu(t) + Mp'(c)q'(c),$$

which has been widely considered in the literature (e.g. see the survey in Sobolev polynomials [?]).

Taking all this into account, the main idea of this paper is to obtain similar results for the polynomials sequence orthogonal with respect to a q-analogue of (**??**), i.e. orthogonal with respect to the Sobolev-type inner product

$$\langle f, g\rangle_S := \langle \mathbf{u}, fg\rangle + N(\mathscr{D}_q f)(\alpha)(\mathscr{D}_q g)(\alpha),$$

where $\mathbf{u}$ is a q-classical linear functional and $N, \alpha \in \mathbb{R}$.

The structure of the paper is the following: In Section 2, we introduce some notation and results we need to prove some of the results throughout the paper. In Section 3, we define the discrete Sobolev-type polynomials and present some algebraic, and analytical, results about these polynomials. And in Section 4 we apply the obtained results to the Al-Salam-Carlitz I polynomials as well as to the Stieltjes–Wigert polynomials.

2 Basic definitions, notations and results

Let $\mathbb{P}$ denote the vector space of univariate, complex-valued, polynomials, and let $\mathbb{P}'$ denote its algebraic dual space. We denote by $\langle \mathbf{u}, p\rangle$ the *duality bracket* for $\mathbf{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$, and $u_n = \langle \mathbf{u}, x^n\rangle$ with $n \geq 0$ are the canonical moments of $\mathbf{u}$.

Definition 1. [?] A linear functional $\mathbf{u}$ is said to be quasi-definite if the principal submatrices of the infinite Hankel matrix associated with the sequence of the moments $(u_n)_n$ of the linear functional $\mathbf{u}$, i.e. $H_k = (u_{i+k})_{i,j=0}^k$, are non-singular.

We are going to consider the quasi-definite linear functional $\mathbf{u}$. Therefore, there exists a sequence of monic polynomials $(p_n)_n$ with $\deg p_n = n$, orthogonal with respect to $\mathbf{u}$, i.e.

$$\langle \mathbf{u}, p_n p_m\rangle = k_n \delta_{n,m}, \qquad k_n = \langle \mathbf{u}, p_n^2\rangle \neq 0.$$

Such a sequence is said to be a monic orthogonal polynomial sequence (MOPS) associated with the linear functional $\mathbf{u}$.

Definition 2. Let $\mathbf{u}$ be a linear functional and let p be a fixed polynomial. We define the linear functional $p\,\mathbf{u}$ as follows:

$$\langle p\,\mathbf{u}, r\rangle := \langle \mathbf{u}, p\,r\rangle, \quad r \in \mathbb{P}.$$

We also need to introduce the concept of quasi-orthogonality that is weaker than the concept of orthogonality.

Definition 3. Given a linear functional $\mathbf{u}$. Let p_n be a polynomial of degree $n \geq r$. If p_n satisfies the conditions

$$\langle \mathbf{u}, x^j p_n\rangle = \begin{cases} 0, & j = 0, 1, \dots, n-r-1, \\ \neq 0, & j = n-r, \end{cases}$$

then p_n is said to be quasi-orthogonal of order r with respect to the linear functional $\mathbf{u}$.

Moreover, if there exists an integral representation of $\mathbf{u}$ as follows

$$\langle \mathbf{u}, p\rangle = \int_I p(x)\, d\mu(x), \quad I \subseteq \mathbb{R},$$

then p_n is said to be quasi-orthogonal of order r with respect to the measure $d\mu(x)$.

The next definitions are related with the q-polynomials located in the Hahn class. In fact, we assume along the paper $0 < q < 1$.

Definition 4. The q-derivative or the Euler–Jackson q-difference operator $\mathscr{D}_q$ is defined as follows:

$$(\mathscr{D}_q f)(x) := \begin{cases} \dfrac{f(qx)-f(x)}{(q-1)x} & \text{if } x \neq 0 \wedge q \neq 1, \\ f'(x) & \text{if } x = 0 \vee q = 1. \end{cases}$$

Definition 5. Given a linear functional $\mathbf{u}$. We say that $\mathbf{u}$ is q-classical if it fulfils the Pearson-type distributional difference equation

$$\mathscr{D}_q[\phi(s)\mathbf{u}] = \psi(s)\mathbf{u}, \quad \phi, \psi \in \mathbb{P},$$

where $\deg\phi \leq 2$, $\deg\psi = 1$. The corresponding MOPS associated with $\mathbf{u}$ is said to be a q-classical discrete MOPS, also called (monic) q-polynomials.

Remark 1. Observe that these functionals $\mathbf{u}$ usually have the form

$$\langle \mathbf{u}, P\rangle = \begin{cases} \displaystyle\int_{s_0}^{s_1} P(x)\rho(x)\, d_q x, & \text{Al-Salam-Carlitz I, discrete } q\text{-Hermite I}, \\ \displaystyle\int_a^b P(x)\rho(x) dx, & \text{Stieltjes–Wigert.} \end{cases}$$

etc., where ρ is a weight function satisfying the following difference equation of Pearson-type

$$\Delta[\phi(s)\rho(s)] = \psi(s)\rho(s)\Delta x(s-1/2) \iff \frac{\rho(s+1)}{\rho(s)} = \frac{\phi(s)+\psi(s)\Delta x(s-1/2)}{\phi(s+1)}.$$

Definition 6. The Jackson q-integrals (see [**?**, **?**]) are defined by

$$\int_0^a f(t)d_q t := a(1-q)\sum_{n=0}^{\infty} f(q^n a)q^n,$$

$$\int_a^0 f(t)d_q t := -a(1-q)\sum_{n=0}^{\infty} f(q^n a)q^n,$$

if $a > 0$ and $a < 0$, respectively. So, we have

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t,$$

and

$$\int_a^b f(t)d_q t = \int_a^0 f(t)d_q t + \int_0^b f(t)d_q t,$$

when $0 < a < b$ and $a < 0 < b$, respectively. Furthermore, we make use of the improper q-Jackson integral

$$\int_0^{\infty} f(t)d_q t := (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

Examples of these polynomials are the big q-Jacobi, the big q-Legendre, the big q-Laguerre, the Little q-Legendre, the Al-Salam-Carlitz I, and the discrete q-Hermite of type I polynomials [**?**, pp. 438, 443, 478, 534, 547], among others.

The next definitions are related with the q-calculus framework:

Definition 7. The q-number $[z]_q$, is defined by

$$[z]_q := \frac{1-q^z}{1-q}, \quad z \in \mathbb{C},$$

Definition 8. A q-analogue of the factorial of n is defined by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \; n = 1, 2, \ldots$$

Definition 9. A q-analogue of the Pochhammer symbol, or shifted factorial, [**?**, **?**] is defined by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \; n = 1, 2, \ldots$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j), \quad |a| < 1.$$

Moreover, we will use the following notation

$$(a_1, \ldots, a_r; q)_k := \prod_{j=1}^{r}(a_j;q)_k.$$

Definition 10. Let $(a_i)_{i=1}^r$ and $(b_j)_{j=1}^s$ be complex numbers such that $b_j \neq q^{-n}$ with $n \in \mathbb{N}_0$, for $j = 1, 2, \ldots, s$. The basic hypergeometric series, or q-hypergeometric, ${}_r\phi_s$ series with variable z is defined by

$${}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k}((-1)^k q^{\binom{k}{2}})^{1+s-r} \frac{z^k}{(q;q)_k}.$$

To complete this section we present some useful results we need along the paper.

Proposition 1. *(Christoffel-Darboux formula). Let (p_n) be a sequence of monic polynomials orthogonal with respect to the linear functional $\mathbf{u}$. If we denote the n-th reproducing kernel by*

$$K_n(x,y) := \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{\langle \mathbf{u}, p_k^2 \rangle}.$$

Then, for all $n \in \mathbb{N}$,

$$K_n(x,y) = \frac{1}{\langle \mathbf{u}, p_{n-1}^2 \rangle} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}. \tag{2}$$

Taking into account the inner product we have considered, then it is natural to consider the partial q-derivatives of $K_n(x,y)$ we will use the following notation:

$$\mathscr{K}_{q,n}^{(i,j)}(x,y) := \sum_{k=0}^{n-1} \frac{(\mathscr{D}_{q,x}^i \, p_k)(x)(\mathscr{D}_{q,y}^j \, p_k)(y)}{\langle \mathbf{u}, p_k^2 \rangle}.$$

The last result we present in this section is a generalization of the reproducing property of the kernel.

Let $\pi(x)$ be a polynomial of degree $n-1$, it can be written in terms of elements of the polynomial sequence $(p_n)_n$, i.e.,

$$\pi(x) = \sum_{k=0}^{n-1} \frac{\langle \mathbf{u}, \pi(x)\, p_k(x) \rangle}{\langle \mathbf{u}, p_k^2 \rangle} p_k(x).$$

Thus, we have

$$(\mathscr{D}^j_{q,y}\pi)(y) = \sum_{k=0}^{n-1} \frac{\langle \mathbf{u}, \pi(x)p_k(x)\rangle}{\langle \mathbf{u}, p_k^2\rangle}(\mathscr{D}^j_q p_k)(y).$$

Then, using the fact that

$$\begin{aligned}\langle \mathbf{u}, \mathscr{K}^{(0,j)}_{q,n}(x,y)\pi(x)\rangle &= \sum_{k=0}^{n-1}\langle \mathbf{u}, \frac{p_k(x)(\mathscr{D}^j_q p_k)(y))}{\langle \mathbf{u}, p_k^2\rangle}\pi(x)\rangle \\ &= \sum_{k=0}^{n-1} \frac{\langle \mathbf{u}, \pi(x)p_k(x)\rangle}{\langle \mathbf{u}, p_k^2\rangle}(\mathscr{D}^j_q p_k)(y).\end{aligned}$$

we obtain the identity

$$\langle \mathbf{u}, \mathscr{K}^{(0,j)}_{q,n}(x,y)\pi(x)\rangle = (\mathscr{D}^j_{q,y}\pi)(y).$$

Observe that for $j = 0$ one has the reproducing property of the kernel, i.e.,

$$\langle \mathbf{u}, K_n(x,y)\pi(x)\rangle = \pi(y).$$

We also need to introduce the following result (see [**?**, Lemma 1] or [**?**, Lemma 3]) about the behaviour of the zeros of a polynomial $f(x) = h_n(x) + cg_n(x)$, that is a linear combination of two polynomials of the same degree.

Lemma 1. *Let $h_n(x) = H(x - x_1)\cdots(x - x_n)$ and $g_n(x) = G(x - y_1)\cdots(x - y_n)$ be two polynomials with real and simple zeros, with $H, G > 0$. If*

$$y_1 < x_1 < \cdots < y_n < x_n,$$

then, for any $c > 0$, the polynomial $f(x) = h_n(x)+cg_n(x)$ has n real zeros, namely $\eta_1 \le \cdots \le \eta_n$, which interlace with the zeros of $h_n(x)$ and $g_n(x)$ as follows

$$y_1 < \eta_1 < x_1 < \cdots < y_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ is a decreasing function of c and, for each $k = 1, \ldots, n$, we have

$$\lim_{c\to\infty} \eta_k(c) = y_k \quad \textit{and} \quad \lim_{c\to\infty} c\big(\eta_k(c) - y_k\big) = \frac{-h_n(y_k)}{g'_n(y_k)}.$$

3 The discrete Sobolev-type polynomials

We start this section introducing the Sobolev-type inner product

$$\langle f, g\rangle_S := \langle \mathbf{u}, fg\rangle + N(\mathscr{D}_q f)(\alpha)(\mathscr{D}_q g)(\alpha), \quad N, \alpha \in \mathbb{R}. \tag{3}$$

We denote by $\left(s_n^{(\alpha)}(N;x)\right)$ the sequence of monic polynomials, orthogonal with respect to the inner product (**??**). These polynomials are said to be basic hypergeometric-Sobolev-type orthogonal polynomials.

3.1 Connection formula

In this section we first express the discrete Sobolev-type polynomials $\left(s_n^{(\alpha)}(N;x)\right)$ in terms of the standard orthogonal polynomials (p_n) and the kernel polynomials and its corresponding derivatives. Taking into account the Fourier expansion, i.e.,

$$s_n^{(\alpha)}(N;x)=p_n(x)+\sum_{k=0}^{n-1}a_{n,k}p_k(x).$$

Then, from the properties of orthogonality of $(p_n)_n$ and $\left(s_n^{(\alpha)}(N;x)\right)_n$ respectively, we have

$$a_{n,k}=\frac{\langle \mathbf{u}, s_n^{(\alpha)}(N;x)p_k(x)\rangle}{\langle \mathbf{u}, p_k^2\rangle}=-\frac{N(\mathscr{D}_q s_n^{(\alpha)})(N;\alpha)(\mathscr{D}_q p_k)(\alpha)}{\langle \mathbf{u}, p_k^2\rangle},\quad 0\le k\le n-1.$$

Thus we deduce

$$s_n^{(\alpha)}(N;x)=p_n(x)-N(\mathscr{D}_q s_n^{(\alpha)})(N;\alpha)\mathscr{K}_{q,n}^{(0,1)}(x,\alpha). \tag{4}$$

Remark 2. Observe that $s_1^{(\alpha)}(N;x)=p_1(x)$.

From here, and after some basic manipulations, one gets

$$(\mathscr{D}_q s_n^{(\alpha)})(N;\alpha)=(\mathscr{D}_q p_n)(\alpha)-N(\mathscr{D}_q s_n^{(\alpha)})(N;\alpha)\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha),$$

Therefore

$$(\mathscr{D}_q s_n^{(\alpha)})(N;\alpha)=\frac{(\mathscr{D}_q p_n)(\alpha)}{1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)},$$

so from (**??**) one has

$$s_n^{(\alpha)}(N;x)=p_n(x)-\frac{N(\mathscr{D}_q p_n)(\alpha)}{1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)}\mathscr{K}_{q,n}^{(0,1)}(x,\alpha). \tag{5}$$

Remark 3. From now on we denote $\frac{N(\mathscr{D}_q p_n)(\alpha)}{1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)}$ by $\mathscr{C}_n$.

Finally taking into account the identity

$$\left(\mathscr{D}_q\frac{f}{x-\alpha}\right)(x)=\frac{(\mathscr{D}_q f)(x)}{qx-\alpha}-\frac{f(x)}{(x-\alpha)(qx-\alpha)},$$

one gets

$$\begin{aligned}\mathscr{K}_{q,n}^{(0,1)}(x,\alpha)=&\frac{p_n(x)p_{n-1}(\alpha)-p_{n-1}(x)p_n(\alpha)}{\|p_{n-1}\|^2\,(x-\alpha)(x-q\alpha)}\\&+\frac{p_n(x)(\mathscr{D}_q p_{n-1})(\alpha)-p_{n-1}(x)(\mathscr{D}_q p_n)(\alpha)}{\|p_{n-1}\|^2\,(x-q\alpha)}.\end{aligned} \tag{6}$$

With this, and by using expression (**??**), one obtains

$$\begin{aligned}s_n^{(\alpha)}(N;x)=p_n(x)-\frac{\mathscr{C}_n}{\|p_{n-1}\|^2}\Bigg(&\frac{p_n(x)p_{n-1}(\alpha)-p_{n-1}(x)p_n(\alpha)}{(x-\alpha)(x-q\alpha)}\\&+\frac{p_n(x)(\mathscr{D}_q p_{n-1})(\alpha)-p_{n-1}(x)(\mathscr{D}_q p_n)(\alpha)}{(x-q\alpha)}\Bigg).\end{aligned} \tag{7}$$

Since the expression (**??**) is a rational function in N, where both the numerator and denominator has the same degree as N, it is clear that we can define the monic polynomial of degree n which results on taking the limit $N \to \infty$. In fact, we obtain

$$r_n^{(\alpha)}(x) := \lim_{N\to\infty} s_n^{(\alpha)}(N;x) = p_n(x) - \frac{(\mathscr{D}_q p_n)(\alpha)}{\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)} \mathscr{K}_{q,n}^{(0,1)}(x,\alpha). \tag{8}$$

To characterize these new polynomials, first we observe that they are strictly quasi-orthogonal of order 2 with respect to the linear functional

$$\mathbf{v} = (x-\alpha)(x-q\alpha)\mathbf{u},$$

therefore, $r_n^{(\alpha)}(x)$ is a linear combination of three consecutive polynomials of the sequence (p_n), i.e., for $n \geq 2$, we have

$$r_n^{(\alpha)}(x) = p_n(x) + b_{q,n} p_{n-1}(x) + c_{q,n} p_{n-2}(x),$$

where

$$c_{q,n} = \frac{\langle \mathbf{u}, r_n^{(\alpha)}(x) p_{n-2}(x) \rangle}{\langle \mathbf{u}, p_{n-2}^2 \rangle} = -\frac{((\mathscr{D}_q p_n)(\alpha))^2}{\mathscr{K}_{q,n}(\alpha,\alpha)} < 0.$$

3.2 Distribution of the zeros

Let $(\eta_{n,k})_{k=1}^n$ be the zeros of $s_n^{(\alpha)}(N;x)$ and $(x_{n,k})_{k=1}^n$ be the zeros of $p_n(x)$. Then the following result holds.

Proposition 2. *([?], Proposition 6.2) The polynomial $r_n^{(\alpha)}(x)$ has n real and simple zeros, namely $(y_{n,k})_{k=1}^n$.*
Moreover, if $\alpha < supp(\mathbf{u})$, then

$$y_{n,1} < \alpha < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < y_{n,n} < x_{n,n},$$

and if $\alpha > supp(\mathbf{u})$, then

$$x_{n,1} < y_{n,1} < x_{n,2} < \cdots < y_{n,n-1} < x_{n,n} < \alpha < y_{n,n},$$

hold for every $n \geq 2$.

In order to obtain some results concerning monotonicity, asymptotics, and speed of convergence for the zeros in terms of the mass N applying Lemma **??** we believe that is more convenient to normalize the connection formula (**??**) in this useful way:

Proposition 3. *For this polynomial sequence the following identity holds:*

$$\left(1 + N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right) s_n^{(\alpha)}(N;x) = p_n(x) + N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha) r_n^{(\alpha)}(x). \tag{9}$$

We leave the proof to the reader, it is just enough to replace (**??**) in (**??**) and after some basic manipulations one gets the desired identity.

We point out the fact the basic hypergeometric-Sobolev-type orthogonal polynomials $s_n^{(\alpha)}(N;x)$ appears as a linear combination of two polynomials of degree n. Thus, from (**??**), Proposition **??**, and Lemma **??**, we immediately conclude that the mass point α does not attract any zero of $s_n^{(\alpha)}(N;x)$ when $N \to \infty$, as in the standard case. By standard we mean the case of the polynomials orthogonal with respect to the inner product (**??**) (see [?]).

Moreover, it is well-known that the polynomial $p_n(x)$ has n different real zeros and since we assumed that the interval $(\alpha, q\alpha)$ contains no points of the spectrum of $\mathbf{u}$, it follows that at most one zero of $p_n(x)$ is situated in $[\alpha, q\alpha]$. For the polynomials $s_n^{(\alpha)}(N;x)$ we have the following result.

Proposition 4. *If $n \geq 3$ the polynomial $s_n^{(\alpha)}(N;x)$ has at least $n-2$ different real zeros with odd multiplicity.*

Proof. Let $\eta_{n,1}, \eta_{n,2}, \cdots, \eta_{n,k}$ denote the real zeros of $s_n^{(\alpha)}(N;x)$ of odd multiplicity. Put $\Pi(x) = (x-\eta_{n,1})\cdots(x-\eta_{n,k})$ We have

$$\langle s_n^{(\alpha)}(N;x), (x-\alpha)(x-q\alpha)\Pi(x)\rangle_S = \langle \mathbf{u}, s_n^{(\alpha)}(N;x)(x-\alpha)(x-q\alpha)\Pi(x)\rangle > 0.$$

Hence $\deg \Pi \geq n-2$. ■

The position of the real zeros of $s_n^{(\alpha)}(N;x)$ can be localized by the following theorem.

Theorem 1. *Suppose $(\mathscr{D}_q p_n)(\alpha) = 0$, $N > 0$. Let k denote the intersection with the real axis of the chord which joins the points $(\alpha, p_n(\alpha))$ and $(q\alpha, p_n(q\alpha))$.*

1. *If $k \notin [x_{n,i}, x_{n,i+1}]$ and $x_{n,i}$, $x_{n,i+1} \notin [\alpha, q\alpha]$, then $(x_{n,i}, x_{n,i+1})$ contains at least one zero of $s_n^{(\alpha)}(N;x)$.*
2. *If there exists a unique $0 \leq i \leq n$ such that $\alpha < x_{n,i} < q\alpha$, then also $\alpha < k < q\alpha$ and we have one of the following cases:*
   - *(a) If $\alpha < k < x_{n,i} < q\alpha$ then $(x_{n,i-1}, x_{n,i})$ contains at least one zero of $s_n^{(\alpha)}(N;x)$.*
   - *(b) If $\alpha < x_{n,i} < k < q\alpha$ then $(x_{n,i}, x_{n,i+1})$ contains at least one zero of $s_n^{(\alpha)}(N;x)$.*

The proof of this result is analogue to the one presented in [**?**, Theorem 3.2] so we leave it to the reader.

Remark 4. Notice that depending on the sign of α the interval changes but the result is clear in both cases $\alpha > 0$ and $\alpha < 0$.

Corollary 1. *The position of at least $n-2$ zeros of $s_n^{(\alpha)}(N;x)$ can be localized.*

Proof. We consider different cases:

- If $[\alpha, q\alpha]$ does not contain a zero of $p_n(x)$, since $x_{n,i} \neq \alpha$ for all i, then for at least $n-2$ intervals $(x_{n,i}, x_{n,i+1})$ part 1. of Theorem **??** can be applied.
- If $(\alpha, q\alpha)$ contains a zero of $p_n(x)$ (more than one is impossible), then $n-3$ zeros of $s_n^{(\alpha)}(N;x)$ can be localized by the first part of Theorem **??** and one by the second part.

Hence the result follows. ■

3.3 The five-term recurrence relation

Since the inner product (**??**) does not satisfy the Hankel property , i.e.,

$$\langle x\, f, g\rangle_S = \langle f, x\, g\rangle_S,$$

the polynomial sequence $\left(s_n^{(\alpha)}(N;x)\right)_n$ does not fulfil a three-term recurrence relation. However, we find that

$$\langle (x-\alpha)(x-\alpha q)p, r\rangle_S = \langle p, (x-\alpha)(x-\alpha q)r\rangle_S, \quad p, r \in \mathbb{P}.$$

Let us state the first result which is a direct consequence of this fact.

Proposition 5. *The following identity holds for $n \geq 2$:*

$$(x-\alpha)(x-\alpha q)s_n^{(\alpha)}(N;x) = \sum_{\nu=n-2}^{n+2} a_{n,\nu}p_\nu(x), \tag{10}$$

where $a_{n,n+2}=1$, and

$$\begin{aligned}
a_{n,n+1} &= \big(\beta_{n+1}+\beta_n-\alpha(1+q)\big) - \frac{\mathscr{C}_n}{\langle \mathbf{u}, p_{n-1}^2\rangle}\left(\mathscr{D}_q p_{n-1}\right)(\alpha),\\
a_{n,n} &= \big(\gamma_{n+1}+\gamma_n+(\beta_{n+1}-\alpha q)(\beta_n-\alpha)\big)\big)\\
&\quad -\frac{\mathscr{C}_n}{\langle \mathbf{u}, p_{n-1}^2\rangle}\big(p_{n-1}(\alpha)-\left(\mathscr{D}_q p_n\right)(\alpha)+(\beta_n-\alpha)\left(\mathscr{D}_q p_{n-1}\right)(\alpha)\big),\\
a_{n,n-1} &= \gamma_n\big(\beta_n+\beta_{n-1}-\alpha(1+q)\big)+\frac{\mathscr{C}_n}{\langle \mathbf{u}, p_{n-1}^2\rangle}\big(p_n(\alpha)-\gamma_n\left(\mathscr{D}_q p_{n-1}\right)(\alpha)\\
&\quad +(\beta_{n-1}-\alpha)\left(\mathscr{D}_q p_n\right)(\alpha)\big),\\
a_{n,n-2} &= \gamma_n\gamma_{n-1}+\frac{\mathscr{C}_n}{\langle \mathbf{u}, p_{n-1}^2\rangle}\left(\mathscr{D}_q p_n\right)(\alpha).
\end{aligned}$$

Indeed, one can obtain all the coefficients of (**??**) by using the three-term recurrence relation of $(p_n)_n$

$$xp_n(x) = p_{n+1}(x)+\beta_n p_n(x)+\gamma_n p_{n-1}(x),$$

with initial conditions $p_{-1}(x)=0$, $p_0(x)=1$, the expression (**??**), the identity

$$a_{n,\nu}\langle u, p_\nu^2\rangle = \langle s_n^{(\alpha)}(N;x),(x-\alpha)(x-\alpha q)p_\nu\rangle_S, \quad \nu=0,1,\ldots,n+2,$$

and using the expansion of $(x-\alpha)(x-\alpha q)p_n(x)$, i.e.

$$\begin{aligned}
(x-\alpha)(x-\alpha q)p_n(x) &= p_{n+2}(x)+\big(\beta_{n+1}+\beta_n-\alpha(1+q)\big)p_{n+1}(x)+\big(\gamma_{n+1}+\gamma_n\\
&\quad +(\beta_{n+1}-\alpha q)(\beta_n-\alpha)\big)p_n(x)+\gamma_n\big(\beta_n+\beta_{n-1}-\alpha(1+q)\big)p_{n-1}(x)\\
&\quad +\gamma_n\gamma_{n-1}p_{n-2}.
\end{aligned}$$

We now derive a recurrence relation for the polynomials $s_n^{(\alpha)}(N;x)$.

Proposition 6. *(Five-term recurrence relation)*
The following recurrence Relation holds for $n \geq 2$:

$$(x-\alpha)(x-\alpha q)s_n^{(\alpha)}(N;x) = \sum_{\nu=n-2}^{n+2} \lambda_{n,\nu}s_\nu^{(\alpha)}(N;x), \tag{11}$$

where $\lambda_{n,n+2}=1$, and

$$\lambda_{n,j} = \frac{a_{j,n}\langle \mathbf{u}, p_n^2\rangle\left(1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right)+N(\mathscr{D}_q p_n)(\alpha)\displaystyle\sum_{i=n+1}^{j+2} a_{j,i}(\mathscr{D}_q p_i)(\alpha)}{(1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha))\|S_j\|_S^2},$$

where

$$\|S_n\|_S^2 := (S_n,S_n)_S = \langle s_n^{(\alpha)}(N;x), s_n^{(\alpha)}(N;x)\rangle_S = \frac{1+N\mathscr{K}_{q,n+1}^{(1,1)}(\alpha,\alpha)}{1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)}\langle \mathbf{u}, p_n^2\rangle.$$

The proof of this result is analogous to the one of the Proposition 3.1 in [?].

Remark 5. Notice that after basic manipulations of (??)we get

$$\lambda_{n,n-2} = \frac{\|S_n\|_S^2}{\|S_{n-2}\|_S^2}, \qquad \lambda_{n,n+1}\|S_{n+1}\|_S^2 = \lambda_{n+1,n}\|S_n\|_S^2.$$

Observe that this result is direct after some straightforward calculations.

3.4 The second order linear q-difference holonomic equation

In the following we assume that $\mathbf{u}$ is a classical q-discrete functional. Let $\Phi(x)$ denote the polynomial $(x-\alpha)(x-\alpha q)$. From the expression (??) we get

$$\Phi(x)s_n^{(\alpha)}(N;x) = A(x;n)p_n(x) + B(x;n)p_{n-1}(x), \tag{12}$$

where $A(x;n)$ and $B(x;n)$ are polynomials of degree bounded by a number independent of n and at most 2 and 1, respectively. On the other hand, since $\mathbf{u}$ is a classical linear functional, then there exist a polynomial $\Psi(x)$ and two polynomials $M(x;n)$ and $N(x;n)$, with degree bounded by a number independent of n, such that

$$\Psi(x)(\mathscr{D}_q p_n)(x) = M(x;n)p_n(x) + N(x;n)p_{n-1}(x). \tag{13}$$

Using (??) and (??) we obtain the following representation formula:

$$\pi(x)s_n^{(\alpha)}(N;x) = a(x;n)p_n(x) + b(x;n)p_n(qx), \tag{14}$$

where a, b and π are polynomials of degree bounded by a number independent of n. With all these expressions we can formulate the result.

Theorem 2. *Let $\mathbf{u}$ be a classical linear functional. Suppose that the polynomials $\left(s_n^{(\alpha)}(N;x)\right)$ are defined by (??) where the polynomial p_n is a solution of a second order linear q-difference holonomic equation, q-SODE in short, of the form*

$$\sigma(x;n)p_n(q^{-1}x) - \varphi(x;n)p_n(x) + \zeta(x;n)p_n(qx) = 0. \tag{15}$$

Then $\left(s_n^{(\alpha)}(N;x)\right)$ satisfy a q-SODE of the form

$$\tilde{\sigma}(x;n)s_n^{(\alpha)}(N;q^{-1}x) - \tilde{\varphi}(x;n)s_n^{(\alpha)}(N;x) + \tilde{\zeta}(x;n)s_n^{(\alpha)}(N;qx) = 0, \tag{16}$$

where $\tilde{\sigma}$, $\tilde{\varphi}$ and $\tilde{\zeta}$ can be computed explicitly.

A completely analogous proof is given in [?, §3.2] so it will be omitted.

Remark 6. It is clear that after some manipulations one can obtain some lowering and raising operators, namely $\mathfrak{a}^\dagger$ and $\mathfrak{a}$. In fact such operators can be written as follows:

$$\mathfrak{a}^\dagger = \mathscr{A}(x;n)\mathscr{D}_q + \mathscr{E}(x;n)I_d, \quad \mathfrak{a} = \mathscr{B}(x;n)\mathscr{D}_{q^{-1}} + \mathscr{F}(x;n)I_d,$$

where I_d represents the identity operator.

4 The examples

4.1 The Al-Salam-Carlitz I polynomials

The Al-Salam-Carlitz I polynomials, ASCI in short, $U_n^{(a)}(x;q)$ were introduced in [?] by Al-Salam and Carlitz (1965) and they are defined via basic hypergeometric series as [?]

$$U_n^{(a)}(x;q) := (-a)^n\, q^{\binom{n}{2}}\, {}_2\phi_1\left(q^{-n}, x^{-1}; 0; q, a^{-1}qx\right). \tag{17}$$

Proposition 7. *For this polynomial sequence the following identity holds:*

1. *Orthogonality relation. For $a<0$*

$$\int_a^1 U_m^{(a)}(x;q)U_n^{(a)}(x;q)\left(qx, a^{-1}qx;q\right)_\infty d_qx = \left\|U_n^{(a)}\right\|^2\delta_{m,n}.$$

where by $\delta_{i,j}$ we denote the Kronecker delta function.

2. *Squared norm.*

$$\left\|U_n^{(a)}\right\|^2 = (1-q)\,(-a)^n\, q^{\binom{n}{2}}\,(q;q)_n\left(q, a, a^{-1}q;q\right)_\infty .$$

3. *The Three-Term Recurrence Relation. For $n\geq 0$,*

$$U_{n+1}^{(a)}(x;q) = (x-(a+1)q^n)U_n^{(a)}(x;q) + aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q), \tag{18}$$

with initial conditions $U_0^{(a)}(x;q)=1$, and $U_{-1}^{(a)}(x;q)=0$.

4. *Forward Shift Operator.*

$$\left(\mathscr{D}_q U_n^{(a)}\right)(x;q) = [n]_q\, U_{n-1}^{(a)}(x;q). \tag{19}$$

4.1.1 Connection formula and hypergeometric representation

We can express the ASCI–Sobolev-type polynomials $\left(U_n^{(a,\alpha)}(N;x;q)\right)$ in terms of the ASCI the associated Kernel polynomials. In fact, by construction, we have

$$U_n^{(a,\alpha)}(N;x;q) = U_n^{(a)}(x;q) - \frac{N[n]_q U_{n-1}^{(a)}(\alpha;q)}{1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)}\mathscr{K}_{q,n}^{(0,1)}(x,\alpha). \tag{20}$$

And by using (??) and (??) we obtain

$$U_n^{(a,\alpha)}(N;x;q) = A_n(x)U_n^{(a)}(x;q) + B_n(x)U_{n-1}^{(a)}(x;q),$$

where

$$A_n(x) = 1 - \frac{N[n]_q U_{n-1}^{(a)}(\alpha;q)\left(U_{n-1}^{(a)}(\alpha;q) + (x-\alpha)[n-1]_q U_{n-2}^{(a)}(\alpha;q)\right)}{\left\|U_{n-1}^{(a)}\right\|^2(x-\alpha)(x-\alpha q)\left(1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right)},$$

and

$$B_n(x) = \frac{N[n]_q U_{n-1}^{(a)}(\alpha;q)\left(U_n^{(a)}(\alpha;q) + (x-\alpha)\,[n]_q U_{n-1}^{(a)}(\alpha;q)\right)}{\left\|U_{n-1}^{(a)}\right\|^2(x-\alpha)(x-\alpha q)\left(1+N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right)}.$$

Thus, taking into account (**??**) as well as the identity

$$\left(q^{1-n};q\right)_k = \frac{q}{[n]_q}\left([n-1]_q - [k-1]_q\right)\left(q^{-n};q\right)_k, \tag{21}$$

we deduce

$$U_n^{(a,\alpha)}(N;x;q) = -(-a)^{n-1} q^{\binom{n}{2}-n+2} \frac{B_n(x)}{[n]_q} \times \sum_{k=0}^{n}\left([k-1]_q + \Theta_n(x)\right)\frac{(q^{-n};q)_k \left(x^{-1};q\right)_k}{(q;q)_k}\left(a^{-1}qx\right)^k,$$

where

$$\Theta_n(x) = \frac{aq^{n-2}[n]_q A_n(x)}{B_n(x)} - [n-1]_q.$$

In addition, we have

$$[k-1]_q + \Theta_n(x) = \frac{1-\varphi_n(x)q^{-1}}{\varphi_n(x)(1-q)}\frac{(\varphi_n(x);q)_k}{(\varphi_n(x)q^{-1};q)_k}, \tag{22}$$

where

$$\varphi_n(x) = \frac{1}{(1-q)\Theta_n(x)+1}.$$

Consequently

$$U_n^{(a,\alpha)}(N;x;q) = -(-a)^{n-1}q^{\binom{n}{2}-n+2}\frac{B_n(x)}{[n]_q}\frac{1-\varphi_n(x)q^{-1}}{\varphi_n(x)(1-q)} \times \sum_{k=0}^{n}\frac{(q^{-n};q)_k\left(x^{-1};q\right)_k(\varphi_n(x);q)_k}{(\varphi_n(x)q^{-1};q)_k}\frac{(a^{-1}qx)^k}{(q;q)_k}.$$

Thus we have proven the following identity for the ASCI-Sobolev-type polynomial.

Theorem 3. *The ASCI-Sobolev-type polynomial has the following hypergeometric representation:*

$$U_n^{(a,\alpha)}(N;x;q) = (-a)^n \frac{B_n(x)(1-\varphi_n(x)q^{-1})q^{\binom{n}{2}-n+2}}{a[n]_q\varphi_n(x)(1-q)} \times {}_3\phi_2\left(q^{-n},x^{-1},\varphi_n(x);0,\varphi_n(x)q^{-1};q,a^{-1}qx\right). \tag{23}$$

4.1.2 Distribution of the zeros

Our main aim in this Section is to study the location as well as to obtain results concerning the monotonicity and speed of convergence of the zeros of the ASCI-Sobolev-type polynomial $U_n^{(a,\alpha)}(N;x;q)$. For this purpose we use Lemma **??**.

Lemma 2. *The polynomial* $\mathscr{K}_{q,n}^{(0,1)}(x,\alpha)$ *has* $n-1$ *real and simple zeros which interlace with the zeros of* $U_n^{(a)}(x;q)$.

The proof is straightforward by applying [**?**, Lemma 1.1] to the expression (**??**) and we leave it to the reader.

Theorem 4. *Taking into account the identity (**??**). For any $\alpha \in \mathbb{R}$, with $\alpha \notin [a,1]$ and for any $N > 0$ the zeros of $U_n^{(a,\alpha)}(N;x;q)$, $U_n^{(a)}(x)$ and $r_n^{(a,\alpha)}(x;q)$ fulfil the following interlacing relations:*

- *If $\alpha < a$ then*

$$y_{n,1} < \eta_{n,1} < x_{n,1} < y_{n,2} < \eta_{n,2} < x_{n,2} < \cdots < y_{n,n} < \eta_{n,n} < x_{n,n}.$$

Moreover, each zero $\eta_{n,k}$ is a decreasing function in N, i.e., $\eta_{n,k} = \eta_{n,k}(N)$; and, for each $k = 2,\ldots,n$

$$\lim_{N\to\infty} \eta_{n,1}(N) = \alpha, \qquad \lim_{N\to\infty} \eta_{n,k}(N) = y_{n,k},$$

and

$$\lim_{N\to\infty} N\Big(\eta_{n,k}(N) - y_{n,k}\Big) = \frac{-U_n^{(a)}(y_{n,k};q)}{\left(r_n^{(a,\alpha)}(x;q)\right)'\Big|_{x=y_{n,k}}}.$$

- *If $\alpha > 1$ then*

$$x_{n,1} < \eta_{n,1} < y_{n,1} < x_{n,2} < \eta_{n,2} < y_{n,2} < \cdots < x_{n,n} < \eta_{n,n} < y_{n,n}.$$

Moreover, each zero $\eta_{n,k}$ is an increasing function in N; and, for each $k = 1,\ldots,n-1$

$$\lim_{N\to\infty} \eta_{n,k}(N) = y_{n,k}, \qquad \lim_{N\to\infty} \eta_{n,n}(N) = \alpha,$$

and

$$\lim_{N\to\infty} N\Big(\eta_{n,k}(N) - y_{n,k}\Big) = \frac{-U_n^{(a)}(y_{n,k};q)}{\left(r_n^{(a,\alpha)}(x;q)\right)'\Big|_{x=y_{n,k}}}.$$

Notice that the mass point α attracts one zero of (**??**), i.e. when $N \to \infty$, it captures either the smallest or the largest zero, according to the location of the point α with respect to $[a,1]$. When either $\alpha < a$ or $\alpha > 1$, at most one of the zeros of (**??**) is located outside of $[a,1]$. Next, we give explicitly the value N_0 of the mass N, such that for $N > N_0$ one of the zeros is located outside $[a,1]$.

Corollary 2. *If $\alpha \notin [a,1]$, the following statements hold:*

i.) if $\alpha < a$, then the smallest zero $\eta_{n,1} = \eta_{n,1}(\alpha)$ satisfies

$$\eta_{n,1} > a, \text{ for } N < N_0,$$
$$\eta_{n,1} = a, \text{ for } N = N_0,$$
$$\eta_{n,1} < a, \text{ for } N > N_0,$$

where $N_0 = N_0(a,n,q,\alpha)$ with

$$N_0 = \left(\frac{[n]_q U_{n-1}^{(a)}(\alpha;q)}{U_n^{(a)}(a;q)} \mathscr{K}_{q,n}^{(0,1)}(a,\alpha) - \mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right)^{-1} > 0.$$

ii.) if $\alpha > 1$, then the largest zero $\eta_{n,n} = \eta_{n,n}(\alpha)$ satisfies

$$\eta_{n,n} < 1, \text{ for } N < N_0,$$
$$\eta_{n,n} = 1, \text{ for } N = N_0,$$
$$\eta_{n,n} > 1, \text{ for } N > N_0,$$

with

$$N_0 = \left(\frac{[n]_q U_{n-1}^{(a)}(\alpha; q)}{U_n^{(a)}(1; q)} \mathscr{K}_{q,n}^{(0,1)}(1, \alpha) - \mathscr{K}_{q,n}^{(1,1)}(\alpha, \alpha) \right)^{-1} > 0.$$

Proof. It suffices to use (**??**) together with the fact that $U_n^{(a,\alpha)}(N; \tau; q) = 0$, with $\tau = a, 1$, if and only if $N = N_0$

$$U_n^{(a,\alpha)}(N; \tau; q) = U_n^{(a)}(\tau; q) - \frac{N_0 \, U_n^{(a)}(\alpha; q)}{1 + N_0 \mathscr{K}_{q,n}^{(1,1)}(\alpha, \alpha)} \mathscr{K}_{q,n}^{(0,1)}(\tau, \alpha) = 0.$$

Thus

$$U_n^{(a)}(\tau; q) = \frac{N_0 \, U_n^{(a)}(\alpha; q)}{1 + N_0 \mathscr{K}_{q,n}^{(1,1)}(\alpha, \alpha)} \mathscr{K}_{q,n}^{(0,1)}(\tau, \alpha).$$

Therefore

$$N_0 = N_0(a, n, q, \alpha) = \left(\frac{[n]_q U_{n-1}^{(a)}(\alpha; q)}{U_n^{(a)}(\tau; q)} \mathscr{K}_{q,n}^{(0,1)}(\tau, \alpha) - \mathscr{K}_{q,n}^{(1,1)}(\alpha, \alpha) \right)^{-1}.$$

■

Notice that, according to the well-known theorem of Hurwitz (see [**?**, **?**]), for n large enough, only one zero of (**??**) is located outside of $[a, 1]$ and it is attracted by α. Next we show some numerical experiments using Wolfram Mathematica software, dealing with the smallest and the largest zero of (**??**). We are interested to show the location and behaviour of these zeros.

In the first two tables we show the position for the first and last zero of (**??**) of degree $n = 20$, $a = -1$ and $q = 1/2$, for some choices of the mass N. Indeed, this case is connected to the discrete $q-$Hermite polynomials [**?**]. For $N = 0$ obviously we recover the first zero and the last zero of (**??**). When the mass point is located at $\alpha = -75 < a$, then $N_0 = 3.12758 \times 10^{-128}$; and when it is located at $\alpha = 72 > 1$, then $N_0 = 1.41561 \times 10^{-127}$ we obtain

N	7.0×10^{-120}	7.0×10^{-118}	7.0×10^{-116}	7.0×10^{-114}
$\eta_{20,1}$	-63.6640	-74.8668	-74.9987	-75.0001
$\eta_{20,20}$	40.5829	71.447	71.9945	72.

The next table shows the smallest and largest zeroes in the case when we set $n = 11$ and $a = -5$, and the mass point is located at $\alpha = -42$ ($N_0 = 8.84157 \times 10^{-41}$) and at $\alpha = 45$ ($N_0 = 6.30851 \times 10^{-41}$) respectively

N	7.0×10^{-36}	7.0×10^{-34}	7.0×10^{-32}	7.0×10^{-30}
$\eta_{11,1}$	-25.9916	-41.7589	-42.0138	-42.0164
$\eta_{11,11}$	41.4986	44.9875	45.0253	45.0257

In the next tables we provide numerical evidences in support of Corollary **??**, where the exact values of N_0 are calculated for the $a = -1$ case. For this purpose we begin by analyzing the smallest zero of (**??**) of degree $n = 14$, $q = 1/2$, and with the mass point located at $\alpha = -3$. In this case we have that $N_0 = N_0(-3, 14, 1/2) = 5.13942 \times 10^{-41}$. Thus

N	5.0×10^{-45}	$N_0\,(-3, 14, 1/2)$	5.0×10^{-35}	5.0×10^{-30}
$\eta_{14,1}$	-0.9999	-1.0000	-2.8906	-3.0001

Finally, for the case where the mass point is located at $\alpha = 21$, we have $N_0\,(21, 14, 1/2) = 4.03796 \times 10^{-62}$ and

N	6.0×10^{-65}	$N_0\,(21, 14, 1/2)$	6.0×10^{-55}	6.0×10^{-50}
$\eta_{14,14}$	0.9999	1.0000	20.7490	21.0013

4.2 The Stieltjes–Wigert polynomials

The (monic) Stieltjes–Wigert polynomials, SW in short, $S_n(x;q)$ are defined via basic hypergeometric series as [**?**]

$$S_n(x;q) := (-1)^n q^{-n^2}\,{}_1\phi_1\left(q^{-n}; 0; q, -q^{n+1}x\right).$$

The moment problem for the SW polynomials is indeterminate; in other words, there are many different measures giving the same family of orthogonal polynomials (Krein's condition, see e.g. [**?**]).

Proposition 8. *For this polynomial sequence the following identities hold:*

1. *Orthogonality relation*

$$\int_0^\infty S_m(x;q)S_n(x;q)\,\frac{dx}{(-x, -qx^{-1}; q)_\infty} = \|S_n\|^2\delta_{m,n}.$$

2. *Squared Norm.*

$$\|S_n\|^2 = -q^{-n(2n+1)}\,(q;q)_\infty(q;q)_n \log q.$$

3. *The recurrence relation. For* $n \geq 0$,

$$S_{n+1}(x;q) = (x - q^{-2n-1}(1 + q - q^{n+1}))S_n(x;q) - q^{-4n+1}(1 - q^n)S_{n-1}(x;q),$$

with initial conditions $S_0(x;q) = 1$, *and* $S_{-1}(x;q) = 0$.

4. *Forward shift operator*

$$\left(\mathscr{D}_q S_n\right)(x;q) = q^{-2(n-1)}[n]_q S_{n-1}(xq^2;q).$$

4.3 Connection formula and hypergeometric representation

We denote by $\left(S_n^{(\alpha)}(N;x;q)\right)$ the sequence of monic polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle_S := \int_0^\infty f(x)g(x)\,\frac{dx}{(-x, -qx^{-1}; q)_\infty} + N\left(\mathscr{D}_q f\right)(\alpha)\left(\mathscr{D}_q g\right)(\alpha).$$

These polynomials are connected with the Stieltjes–Wigert polynomials by the formula

$$S_n^{(\alpha)}(N;x;q) = S_n(x;q) - \frac{N(\mathscr{D}_q S_n)(\alpha)}{1 + N\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)}\mathscr{K}_{q,n}^{(0,1)}(x,\alpha). \tag{24}$$

Next, using the same idea than with the another example and by using the identities (**??**) and (**??**), we get the following result.

Theorem 5. *The SW-Sobolev-type polynomial has the following hypergeometric representation*

$$S_n^{(\alpha)}(N;x;q)=\frac{B_n(x)(1-\varphi_n(x)q^{-1})}{(q;q)_n(q^{-n}-1)}\,{}_2\phi_2\left(q^{-n},\varphi_n(x);0,\varphi_n(x)q^{-1};q,-q^nx\right),$$

where, in this case,

$$\varphi_n(x)=\frac{B_n(x)-A_n(x)(q^n-1)}{B_n(x)}\,q^{1-n}.$$

4.3.1 Distribution of the zeros

As in the previous example our main aim, again, in this Section is to study the location as well as to obtain results concerning the monotonicity and speed of convergence of the zeros of the Stieltjes–Wigert-Sobolev-type polynomial, $S_n^{(\alpha)}(N;x;q)$.

Notice that, when $\alpha\in\mathbb{R}$, with $\alpha<0$, at most one of the zeros of $S_n^{(\alpha)}(N;x;q)$ is located outside $[0,\infty)$. Next we provide the explicit value N_0 of the mass such that for $N>N_0$ this situation appears, i.e, one of the zeros is located outside $[0,\infty)$.

Theorem 6. *Taking into account the identity* (**??**). *For any $\alpha\in\mathbb{R}$, with $\alpha<0$ and for any $N>0$ the zeros of $S_n^{(\alpha)}(N;x;q)$, $S_n(x;q)$ and $r_n^{(\alpha)}(x;q)$ fulfil the following interlacing relations:*

$$y_{n,1}<\eta_{n,1}<x_{n,1}<y_{n,2}<\eta_{n,2}<x_{n,2}<\cdots<y_{n,n}<\eta_{n,n}<x_{n,n}.$$

Moreover, each zero $\eta_{n,k}$ is a decreasing function in N, i.e., $\eta_{n,k}=\eta_{n,k}(N)$; and, for each $k=2,\ldots,n$

$$\lim_{N\to\infty}\eta_{n,1}(N)=\alpha,\qquad \lim_{N\to\infty}\eta_{n,k}(N)=y_{n,k},$$

and

$$\lim_{N\to\infty}N\Big(\eta_{n,k}(N)-y_{n,k}\Big)=\frac{-S_n(y_{n,k};q)}{\left(r_n^{(\alpha)}(x;q)\right)'\Big|_{x=y_{n,k}}}.$$

Corollary 3. *If $\alpha\in\mathbb{R}$, with $\alpha<0$, then the smallest zero $\eta_{n,1}=\eta_{n,1}(\alpha)$ satisfies*

$$\eta_{n,1}>0,\ \text{for } N<N_0,$$
$$\eta_{n,1}=0,\ \text{for } N=N_0,$$
$$\eta_{n,1}<0,\ \text{for } N>N_0,$$

where

$$N_0=N_0(\alpha,n,q)=\left(\frac{\mathscr{D}_qS_n(\alpha;q)}{S_n(0;q)}\mathscr{K}_{q,n}^{(0,1)}(0,\alpha)-\mathscr{K}_{q,n}^{(1,1)}(\alpha,\alpha)\right)^{-1}>0.$$

The proof of this result is analogue to the previous example so we leave it to the reader. The numerical behaviour of the zeros for this family is very similar to the Al-Salam-Carlitz case, hence we are going to show here how changes the smallest zero of degree $n=14$ when $\alpha=-1$ for some real values of $0<q\le 1$ going to 1.

q	$N_0\,(21,14,1/2)$	N	$10^{-5}N_0$	N_0	10^5N_0	$10^{10}N_0$
0.5	0.2717	$\eta_{14,1}$	1.2482	0.0000	−3.2380	−3.2381
0.8	2.95394×10^{-8}	$\eta_{14,1}$	0.5574	0.0000	−1.4878	−1.4878
0.9	8.29261×10^{-17}	$\eta_{14,1}$	0.4657	0.0000	−1.2625	−1.2625
0.99	1.42958×10^{-97}	$\eta_{14,1}$	0.6206	0.0000	−1.1597	−1.1597

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