

# On the relation between Gegenbauer polynomials and the Ferrers function of the first kind

Howard S. Cohl<sup>†</sup> and Roberto S. Costas-Santos<sup>§</sup>

<sup>†</sup> *Applied and Computational Mathematics Division, National Institute of Standards and Technology, Mission Viejo, CA 92694, USA*

URL: <http://www.nist.gov/itl/math/msg/howard-s-cohl.cfm>

E-mail: [howard.cohl@nist.gov](mailto:howard.cohl@nist.gov)

<sup>§</sup> *Dpto. de Física y Matemáticas, Universidad de Alcalá, c.p. 28871, Alcalá de Henares, Spain*

URL: <http://www.rscosan.com>

E-mail: [rscosa@gmail.com](mailto:rscosa@gmail.com)

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**Abstract.** Using the direct relation between the Gegenbauer polynomials  $C_n^\lambda(x)$  and the Ferrers function of the first kind  $P_\nu^\mu(x)$ , we compute interrelations between certain Jacobi polynomials, Meixner polynomials, and Ferrers functions of the first and second kind. We then compute Rodrigues-type, standard integral orthogonality and Sobolev orthogonality relations for Ferrers functions of the first and second kinds. In the remainder of the paper using the relation between Gegenbauer polynomials and the Ferrers function of the first kind we derive connection and linearization relations, some definite integral and series expansions, Christoffel-Darboux summation formulas, Poisson kernel and infinite series closure relations (Dirac delta distribution expansions).

*Key words:* Ferrers functions; Gegenbauer polynomials; orthogonal polynomials; orthogonality relations; Christoffel-Darboux summation; Poisson kernel; Closure relations

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## 1 Introduction

The generalized hypergeometric function [8, Chapter 16] is defined by the infinite series [8, (16.2.1)]

$${}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r)_k z^k}{(b_1, \dots, b_s)_k k!}, \quad (1)$$

where  $b_i \notin -\mathbb{N}_0$ , for  $i = 1, \dots, s$ ; and elsewhere by analytic continuation. One interesting limit which we will use below is [12, (1.4.4)]

$$\lim_{\lambda \rightarrow \infty} {}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{s-1}, \lambda b_s \end{matrix}; \lambda z \right) = {}_rF_{s-1} \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{s-1} \end{matrix}; \frac{z}{b_s} \right). \quad (2)$$

The Pochhammer symbol for  $a \in \mathbb{C}$ ,  $n \in \mathbb{N}_0$  is given by [8, (5.2.4-5)]  $(a)_n := (a)(a+1)\cdots(a+n-1)$ . The gamma function [8, Chapter 5] is related to the Pochhammer symbol, namely for  $a \in \mathbb{C} \setminus -\mathbb{N}_0$ , one has  $(a)_n = \Gamma(a+n)/\Gamma(a)$ , which allows one to extend the definition to non-positive integer values of  $n$ . We will also use the common notational product convention, e.g.,  $(a_1, \dots, a_r)_k := (a_1)_k (a_2)_k \cdots (a_r)_k$ .

The Jacobi polynomial is defined as [8, (18.5.7)]

$$P_n^{(\alpha, \beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1 \left( \begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right), \quad (3)$$

and the Gegenbauer function of the first kind is defined as [8, (15.9.15)] (see also [9])

$$C_\alpha^\lambda(z) := \frac{\Gamma(2\lambda + \alpha)}{\Gamma(2\lambda)\Gamma(\alpha + 1)} {}_2F_1\left(\begin{matrix} -\alpha, 2\lambda + \alpha \\ \lambda + \frac{1}{2} \end{matrix}; \frac{1-z}{2}\right), \quad (4)$$

and the classical orthogonal Gegenbauer (ultraspherical) polynomial is given when  $\alpha \in \mathbb{N}_0$ , which makes the Gauss hypergeometric function terminating. Note that the Gegenbauer polynomial can be given in terms of the symmetric Jacobi polynomial as [8, (18.7.1)]

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x). \quad (5)$$

Ferrers functions and Legendre functions are given in terms of Gauss hypergeometric functions which satisfy both linear and quadratic transformations. There are many such transformations and therefore there are many hypergeometric representations for the Gauss hypergeometric function. The Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut)  $P_\nu^{-\mu} : (-1, 1) \rightarrow \mathbb{C}$  can for instance be defined as [8, (14.3.1)]

$$P_\nu^{-\mu}(x) = \frac{1}{\Gamma(\mu + 1)} \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} {}_2F_1\left(\begin{matrix} -\nu, \nu + 1 \\ \mu + 1 \end{matrix}; \frac{1-x}{2}\right). \quad (6)$$

The Ferrers function of the second kind  $Q_\nu^\mu : (-1, 1) \rightarrow \mathbb{C}$  can also be defined in terms of Gauss hypergeometric representations. See [6] where all Gauss hypergeometric representations of the Ferrers functions of the second kind are given. The Ferrers function of the second kind (associated Legendre function of the second kind on-the-cut) for instance, is given in [8, (14.3.2)]

$$\begin{aligned} Q_\nu^\mu(x) := & \frac{\pi}{2 \sin(\pi\mu)} \left( \frac{\cos(\pi\mu)}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} {}_2F_1\left(\begin{matrix} -\nu, \nu + 1 \\ 1-\mu \end{matrix}; \frac{1-x}{2}\right) \right. \\ & \left. - \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1+\mu)} {}_2F_1\left(\begin{matrix} -\nu, \nu + 1 \\ 1+\mu \end{matrix}; \frac{1-x}{2}\right) \right), \quad (7) \end{aligned}$$

where  $\mu \notin \mathbb{Z}$ . However,  $Q_\nu^\mu$  can be analytically continued for  $\mu \in \mathbb{Z}$  which is demonstrated by [8, (14.3.12)]. Note that in this paper we will often indicate that the domain of  $P_\nu^\mu$  and  $Q_\nu^\mu$  is  $(-1, 1)$ . But it should be emphasized that, in general and for specific formulas, this region can be analytically continued to a much larger region in the complex plane. It will often be convenient to express the Ferrers function of the first kind in terms of a Gauss hypergeometric series in a different way. One powerful such series representation is given as follows.

**Lemma 1.1.** *Let  $\nu, \mu \in \mathbb{C}$ ,  $x \in (-1, 1)$ . Then*

$$P_\nu^{-\mu}(x) = \frac{x^{\nu-\mu}(1-x^2)^{\frac{\mu}{2}}}{2^\mu \Gamma(\mu + 1)} {}_2F_1\left(\begin{matrix} \frac{\mu-\nu}{2}, \frac{\mu-\nu+1}{2} \\ \mu + 1 \end{matrix}; 1 - \frac{1}{x^2}\right). \quad (8)$$

**Proof.** Applying the connection relation [8, (14.9.1)] to the Gauss hypergeometric representation of  $Q_\nu^\mu$  [4, (49)]

$$\begin{aligned} Q_\nu^\mu(x) = & \frac{2^{\mu-1} \cos(\pi\mu)}{(1-x^2)^{\frac{\mu}{2}}} \Gamma(\mu) x^{\nu+\mu} {}_2F_1\left(\begin{matrix} \frac{-\nu-\mu}{2}, \frac{-\nu-\mu+1}{2} \\ 1-\mu \end{matrix}; 1 - \frac{1}{x^2}\right) \\ & + \frac{\Gamma(\nu + \mu + 1)\Gamma(-\mu)}{2^{\mu+1}\Gamma(\nu - \mu + 1)} (1-x^2)^{\frac{\mu}{2}} x^{\nu-\mu} {}_2F_1\left(\begin{matrix} \frac{\mu-\nu}{2}, \frac{\mu-\nu+1}{2} \\ \mu + 1 \end{matrix}; 1 - \frac{1}{x^2}\right), \end{aligned}$$

and after some simplification, one completes the proof. ■

One special case which we will encounter frequently below is as follows.

**Corollary 1.2.** *Let  $n \in \mathbb{N}_0$ ,  $x \in (-1, 1)$ . Then*

$$P_{n+\lambda}^{-\lambda}(x) = \frac{x^n(1-x^2)^{\frac{\lambda}{2}}}{2^\lambda \Gamma(\lambda+1)} {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1+\lambda \end{matrix}; 1 - \frac{1}{x^2}\right) = \frac{n!(1-x^2)^{\frac{\lambda}{2}}}{2^\lambda \Gamma(\lambda+1)(2\lambda+1)_n} C_n^{\lambda+\frac{1}{2}}(x). \quad (9)$$

**Proof.** Letting  $\mu \mapsto \lambda$ , and  $\nu \mapsto \lambda+n$  in (8) proves the first relation. The second identity holds due to the definition of the Gegenbauer function of the first kind (4) for  $\alpha \in \mathbb{N}_0$ . This completes the proof.  $\blacksquare$

Some other special cases which are interesting and useful are [8, (14.5.11–14)]

$$P_\nu^{\frac{1}{2}}(\cos \theta) = \sqrt{\frac{2}{\pi \sin \theta}} \cos\left(\left(\nu + \frac{1}{2}\right)\theta\right), \quad Q_\nu^{\frac{1}{2}}(\cos \theta) = -\sqrt{\frac{\pi}{2 \sin \theta}} \sin\left(\left(\nu + \frac{1}{2}\right)\theta\right), \quad (10)$$

$$P_\nu^{-\frac{1}{2}}(\cos \theta) = \sqrt{\frac{2}{\pi \sin \theta}} \frac{\sin\left(\left(\nu + \frac{1}{2}\right)\theta\right)}{\nu + \frac{1}{2}}, \quad Q_\nu^{-\frac{1}{2}}(\cos \theta) = \sqrt{\frac{\pi}{2 \sin \theta}} \frac{\cos\left(\left(\nu + \frac{1}{2}\right)\theta\right)}{\nu + \frac{1}{2}}. \quad (11)$$

Using the connection relation [8, (14.9.1)],  $P_\nu^\mu$  can be expressed in terms of  $Q_\nu^\mu$ , namely

$$P_{k+n-\frac{1}{2}}^{\frac{1}{2}-n}(x) = \frac{2(-1)^n k!}{\pi(2n+k-1)!} Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x). \quad (12)$$

The Ferrers function of the first kind  $P_\nu^{-\mu}$  is related to the Gegenbauer function of the first kind [8, (14.3.21)]

$$P_\nu^{-\mu}(x) = \frac{\Gamma(2\mu+1)\Gamma(\nu-\mu+1)}{2^\mu \Gamma(\nu+\mu+1)\Gamma(\mu+1)} (1-x^2)^{\frac{\mu}{2}} C_{\nu-\mu}^{\mu+\frac{1}{2}}(x) \quad (13)$$

where  $2\mu+1, \nu-\mu+1 \notin -\mathbb{N}_0$ . Equivalently

$$C_\lambda^\mu(x) = \frac{\sqrt{\pi} \Gamma(2\mu+\lambda)}{2^{\mu-\frac{1}{2}} \Gamma(\mu)\Gamma(\lambda+1)} \frac{P_{\lambda+\mu-\frac{1}{2}}^{\frac{1}{2}-\mu}(x)}{(1-x^2)^{\frac{\mu}{2}-\frac{1}{4}}}, \quad (14)$$

or with  $\lambda = n \in \mathbb{N}_0$ , then

$$C_n^\mu(x) = \frac{\sqrt{\pi} \Gamma(2\mu+n)}{2^{\mu-\frac{1}{2}} \Gamma(\mu)n!} \frac{P_{n+\mu-\frac{1}{2}}^{\frac{1}{2}-\mu}(x)}{(1-x^2)^{\frac{\mu}{2}-\frac{1}{4}}}. \quad (15)$$

**Remark 1.3.** *There are also interrelations between certain Jacobi polynomials and the Ferrers function of the first kind from Gegenbauer polynomials using a quadratic transformations, namely for  $n \in \mathbb{N}_0$ ,  $\lambda \in \mathbb{C}$ :*

$$P_n^{(\lambda-\frac{1}{2}, -\frac{1}{2})}(2x^2-1) = \frac{2^\lambda \Gamma(\lambda+n+1)}{n!(1-x^2)^{\frac{\lambda}{2}}} P_{2n+\lambda}^{-\lambda}(x), \quad P_n^{(\lambda, \frac{1}{2})}(2x^2-1) = \frac{2^\lambda \Gamma(\lambda+n+1)}{n!x(1-x^2)^{\frac{\lambda}{2}}} P_{2n+\lambda+1}^{-\lambda}(x). \quad (16)$$

We leave the derivation of these simple identities which follow from [8, (18.7.15)] and [8, (18.7.16)] to the reader.

Below, the following related functions will be used. The associated Legendre function of the first kind  $P_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C}$  is defined as [8, (14.3.6)]

$$P_\nu^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^\mu \Gamma(\mu + 1)} {}_2F_1\left(\begin{matrix} \nu + \mu + 1, -\nu + \mu \\ 1 + \mu \end{matrix}; \frac{1 - z}{2}\right). \quad (17)$$

Note that the Legendre polynomials are given by [8, (14.7.1), (18.7.9)]

$$P_n(x) = P_n^0(x) = P_n^0(x) = C_n^{\frac{1}{2}}(x). \quad (18)$$

**Remark 1.4.** *The Chebyshev polynomials of the first kind [12, (9.8.35)] can be given by*

$$T_n(\cos \theta) := \cos(n\theta) = \sqrt{\frac{\pi \sin \theta}{2}} P_{n-\frac{1}{2}}^{\frac{1}{2}}(\cos \theta) = n \sqrt{\frac{2 \sin \theta}{\pi}} Q_{n-\frac{1}{2}}^{-\frac{1}{2}}(\cos \theta), \quad (19)$$

and the Chebyshev polynomials of the second kind [12, (9.8.36)] can be given by

$$U_n(\cos \theta) := \frac{\sin((n+1)\theta)}{\sin \theta} = (n+1) \sqrt{\frac{\pi}{2 \sin \theta}} P_{n+\frac{1}{2}}^{-\frac{1}{2}}(\cos \theta) = -\sqrt{\frac{2}{\pi \sin \theta}} Q_{n+\frac{1}{2}}^{\frac{1}{2}}(\cos \theta), \quad (20)$$

where we have used (10), (11). See Remark 2.8 below with a concrete orthogonality example where the prefactor  $n$  cancels for the Chebyshev polynomials of the first kind, even for  $n = 0$ .

The associated Legendre function of the second kind  $Q_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C}$  can be defined in terms of the Gauss hypergeometric function as [8, (14.3.10) and Section 14.21]

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}(z^2 - 1)^{\mu/2}}{2^{\nu+1} \Gamma(\nu + \frac{3}{2}) z^{\nu+\mu+1}} {}_2F_1\left(\begin{matrix} \frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2} \\ \nu + \frac{3}{2} \end{matrix}; \frac{1}{z^2}\right), \quad (21)$$

for  $|z| > 1$  and, by analytic continuation of the Gauss hypergeometric function, elsewhere on  $z \in \mathbb{C} \setminus (-\infty, 1]$ . The normalized notation  $Q_\nu^\mu(z)$  is due to Olver [8, (14.3.10)] and is defined in terms of the more commonly appearing Hobson notation for the associated Legendre function of the second kind  $Q_\nu^\mu(z)$  as [8, (14.3.10)]  $Q_\nu^\mu(z) =: e^{i\pi\mu} \Gamma(\nu + \mu + 1) Q_\nu^\mu(z)$ .

The Meixner polynomials are defined by [12, (9.10.1)]  $M_n(x; \beta, c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c}\right)$ , where  $n \in \mathbb{N}_0$ ,  $c \in \mathbb{C} \setminus \{0, 1\}$ . Specialized Meixner polynomials are related to  $P_\nu^\mu$ .

**Theorem 1.5.** *Let  $n \in \mathbb{N}_0$ ,  $\lambda \in \mathbb{C}$ ,  $x \in (-1, 1)$ . Then*

$$P_{n+\lambda}^{-\lambda}(x) = \frac{(1-x^2)^{\frac{\lambda}{2}}}{2^\lambda \Gamma(\lambda+1)} M_n\left(-2\lambda - n - 1; \lambda + 1, \frac{2}{1+x}\right). \quad (22)$$

**Proof.** Start with the relation between the Meixner polynomials and the Jacobi polynomials [12, p. 236]  $M_n(x; \beta, c) = \frac{n!}{(\beta)_n} P_n^{(\beta-1, -\beta-n-x)}((2-c)/c)$ , and let  $x = 1 - n - 2\beta$ . This converts the Jacobi polynomial to symmetric form and then using (5) with  $\beta = \lambda + \frac{1}{2}$ , one obtains  $C_n^\lambda(\omega) = \frac{(2\lambda)_n}{n!} M_n\left(-2\lambda - n; \lambda + \frac{1}{2}, \frac{2}{1+\omega}\right)$ . Finally, using (15), completes the proof. ■

## 1.1 Rodrigues-type relations

From the Gegenbauer polynomial Rodrigues-type relations, we can derive Rodrigues-type relations for the Ferrers function of the first and second kinds.

**Lemma 1.6.** *Let  $x \in (-1, 1)$ ,  $\mu \in \mathbb{C}$ ,  $n \in \mathbb{N}_0$ . Then,*

$$P_{n+\mu}^{-\mu}(x) = \frac{(-1)^n}{2^{\mu+n} \Gamma(\mu+n+1) (1-x^2)^{\frac{\mu}{2}}} \left(\frac{d}{dx}\right)^n \left[(1-x^2)^{\mu+n}\right]. \quad (23)$$

**Proof.** Start with (13), setting  $\nu \rightarrow \mu + n$  and by using the Rodrigues-type formula for the Gegenbauer polynomials [8, (18.5.5)] and after straightforward calculation completes the proof. ■

Similarly, we have a Rodrigues-type relation for  $Q_\nu^\mu$ .

**Corollary 1.7.** *Let  $x \in (-1, 1)$ ,  $\mu \in \mathbb{C}$ ,  $n \in \mathbb{N}_0$ . Then,*

$$Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) = \frac{\pi(-1)^{n+k}(2n+k-1)!}{2^{k+n+\frac{1}{2}}k!\Gamma(k+n+\frac{1}{2})(1-x^2)^{\frac{n}{2}-\frac{1}{4}}} \left(\frac{d}{dx}\right)^k \left[(1-x^2)^{k+n-\frac{1}{2}}\right]. \quad (24)$$

**Proof.** Start with (23), let  $n \mapsto k$ ,  $\mu \mapsto n - \frac{1}{2}$  then use (12), and after simplification, this completes the proof. ■

## 2 Orthogonality relations

Let us consider a complete set of orthogonal functions  $(\psi_n(x))_n$  on the real interval  $(a, b)$  with a weight function  $w(x)$ , with respect to an inner product defined as

$$\langle f, g \rangle := \int_a^b f(x)g(x)w(x) dx. \quad (25)$$

Then, the integral orthogonality relation is given by

$$\langle \psi_n, \psi_m \rangle = \int_a^b \psi_n(x)\psi_m(x)w(x) dx = h_n \delta_{n,m}, \quad n, m = 0, 1, 2, \dots, \quad (26)$$

where  $\delta_{n,m}$  is the Kronecker delta symbol, the constant  $h_n = \langle \psi_n, \psi_n \rangle$ , is the norm of orthogonality with respect to the inner product  $\langle \bullet, \bullet \rangle$ .

In this section, we derive several new forms of orthogonality for Ferrers functions of the first and second kinds which are equivalent to the orthogonality of Gegenbauer (ultraspherical) polynomials. Due to this equivalence, it is surprising that these orthogonality relations have not been noticed previously. We will use these orthogonality conditions to verify our derived eigenfunctions for the Laplace-Beltrami operator on the  $d$ -dimensional  $R$ -radius hypersphere  $\mathbb{S}_R^d$ . For detailed information about the special functions we use below, namely the gamma function  $\Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \rightarrow \mathbb{C}$ , Ferrers functions  $P_\nu^\mu$ ,  $Q_\nu^\mu$ , and the Gegenbauer polynomials  $C_n^\mu$ , and their properties, see [8, Chapters 5, 14, 18].

### 2.1 Continuous orthogonality from Gegenbauer polynomials

**Theorem 2.1.** *Let  $\mu \in \mathbb{C}$ ,  $k, k' \in \mathbb{N}$ ,  $\mu + k + \frac{1}{2} \neq 0$ ,  $\Re\mu > -1$ . Then,*

$$\int_{-1}^1 P_{k+\mu}^{-\mu}(x)P_{k'+\mu}^{-\mu}(x) dx = \frac{k!}{\Gamma(2\mu+k+1)(\mu+k+\frac{1}{2})} \delta_{k,k'}. \quad (27)$$

**Proof.** Combining [8, (14.3.21)] (13) and orthogonality for the Gegenbauer polynomials [8, Sections 18.2-3]

$$\int_{-1}^1 C_n^\mu(x)C_{n'}^\mu(x)(1-x^2)^{\mu-\frac{1}{2}} dx = \frac{\pi\Gamma(2\mu+n)}{2^{2\mu-1}(\mu+n)n!(\Gamma(\mu))^2} \delta_{n,n'}, \quad (28)$$

where  $n, n' \in \mathbb{N}_0$ ,  $\mu \in (-\frac{1}{2}, \infty) \setminus \{0\}$ , with  $\nu = k + \mu$  (resp.  $\nu = k' + \mu$ ) produces (27). The restriction on  $\Re\mu$  comes from ensuring that the singularities at  $x = \pm 1$  are integrable. This completes the proof. ■

Now specializing the above orthogonality relation using  $\mu$  as either an integer or an odd-half-integer produces other orthogonality relations as corollaries.

**Corollary 2.2.** *Let  $n, k, k' \in \mathbb{N}_0$ . Then*

$$\int_{-1}^1 P_{k+n}^n(x) P_{k'+n}^n(x) dx = \frac{(2n+k)!}{k!(n+k+\frac{1}{2})} \delta_{k,k'}. \quad (29)$$

**Proof.** Let  $\mu = n \in \mathbb{N}_0$  in (27), and the connection relation [8, (14.9.3)] expresses the Ferrers functions of the first kind with negative integer order in terms of Ferrers functions of the first kind with positive integer order. ■

Similarly we can derive orthogonality with non-positive integer order.

**Remark 2.3.** *Starting from Theorem 2.1, we can also derive a similar limiting result*

$$\int_{-1}^1 P_{k+n}^{-n}(x) P_{k'+n}^{-n}(x) dx = \frac{k!}{(2n+k)!(n+k+\frac{1}{2})} \delta_{k,k'}. \quad (30)$$

**Corollary 2.4.** *Let  $n, k, k' \in \mathbb{N}_0$ . Then*

$$\int_{-1}^1 P_{k+n}^n(x) P_{k'+n}^{-n}(x) dx = \frac{(-1)^n}{n+k+\frac{1}{2}} \delta_{k,k'}. \quad (31)$$

**Proof.** Using the connection relation [8, (14.9.3)] once with (29) completes the proof. ■

**Remark 2.5.** *The orthogonality relations (27), (29) generalize [8, (14.17.6)],*

$$\int_{-1}^1 P_k^l(x) P_{k'}^l(x) dx = \frac{(k+l)!}{(k-l)!(k+\frac{1}{2})} \delta_{k,k'}, \quad (32)$$

by setting  $n \mapsto n+l$ , and taking  $n=0$  in (29). Similarly, the orthogonality relation (31) generalizes [8, (14.17.7)],

$$\int_{-1}^1 P_k^l(x) P_{k'}^{-l}(x) dx = \frac{(-1)^l}{k+\frac{1}{2}} \delta_{k,k'}, \quad (33)$$

by setting  $n=0$ .

One also has the following orthogonality relation for the Ferrers function of the second kind.

**Corollary 2.6.** *Let  $n, k, k' \in \mathbb{N}_0$ . Then*

$$\int_{-1}^1 Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) Q_{k'+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) dx = \frac{\pi^2(2n+k-1)!}{4(n+k)k!} \delta_{k,k'}. \quad (34)$$

**Proof.** Let  $\mu = n - \frac{1}{2}$ ,  $n \in \mathbb{N}_0$  in (27), and the connection relation [8, (14.9.1)] expresses the Ferrers functions of the first kind with negative integer order in terms of Ferrers functions of the second kind with positive integer order. ■

**Corollary 2.7.** *Let  $l, k, k' \in \mathbb{Z}$ ,  $l+k+1 \geq 0$ ,  $k+1 \neq 0$ ,  $l \geq -1$ . Then*

$$\int_{-1}^1 Q_{k+\frac{1}{2}}^{l+\frac{1}{2}}(x) Q_{k'+\frac{1}{2}}^{l+\frac{1}{2}}(x) dx = \frac{\pi^2(k+l+1)!}{4(k-l)!(k+1)} \delta_{k,k'}. \quad (35)$$

**Proof.** Specializing (34) with  $n = 0$  completes the proof.  $\blacksquare$

**Remark 2.8.** For  $l \in \{-1, 0\}$ , (35) reduces to orthogonality for trigonometric functions, or Chebyshev polynomials of the first and second kinds. For  $l = -1$ , (35) reduces to orthogonality for Chebyshev polynomials of the first kind (19), namely [12, (9.8.37)]

$$\int_0^\pi \cos(k\theta) \cos(k'\theta) d\theta = \int_{-1}^1 T_k(x) T_{k'}(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\epsilon_k} \delta_{k,k'}, \quad (36)$$

where  $\epsilon_k = 2 - \delta_{k,0}$  is the Neumann factor. For  $l = 0$ , (35) reduces to

$$\int_0^\pi \sin(k\theta) \sin(k'\theta) d\theta = \delta_{k,k'} \begin{cases} 0, & \text{if } k = 0, \\ \frac{\pi}{2}, & \text{if } k \geq 1, \end{cases} \quad (37)$$

or for  $k \geq 0$ , [12, (9.8.38)]

$$\int_{-1}^1 \sin((k+1)\theta) \sin((k'+1)\theta) d(\cos \theta) = \int_{-1}^1 U_k(x) U_{k'}(x) \sqrt{1-x^2} dx = \frac{\pi}{2} \delta_{k,k'}. \quad (38)$$

## 2.2 Sobolev orthogonality from Gegenbauer polynomials

Motivated by [1] and [14], we are led to the following result which follows by using the identity (9), the recurrence relations for the Gegenbauer polynomials [12, (9.8.21)], and the property of orthogonality of the Gegenbauer polynomials (28). In order to compute the Sobolev orthogonality which arises from Gegenbauer polynomials in terms of Ferrers functions we will require several particular interrelations between Gegenbauer polynomials in terms of Ferrers functions.

**Lemma 2.9.** Let  $x \in (-1, 1)$ ,  $n, N \in \mathbb{N}_0$ . Then

$$C_n^{\frac{1}{2}-N}(x) = 2^N \left(\frac{1}{2}\right)_N S_{n,N}(x), \quad (39)$$

where

$$S_{n,N}(x) := (1-x^2)^{\frac{N}{2}} \begin{cases} P_{n-N}^{-N}(x) + \frac{(-1)^{n+N}}{n!(2N-n-1)!} Q_{n-N}^N(x), & 0 \leq n \leq 2N-1, \\ P_{n-N}^{-N}(x), & n \geq 2N. \end{cases} \quad (40)$$

**Proof.** The relation [8, (14.9.7)] produces

$$P_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda}(x) = \frac{n!}{2 \sin(\pi\lambda) \Gamma(2\lambda+n)} \left( P_{n+\lambda-\frac{1}{2}}^{\lambda-\frac{1}{2}}(x) + (-1)^n P_{n+\lambda-\frac{1}{2}}^{\lambda-\frac{1}{2}}(-x) \right). \quad (41)$$

Then applying (41) to (15) with  $\lambda = \frac{1}{2} - N$  and [8, (14.9.9)] to  $P_\nu^\mu(-x)$  with simplification produces the result for  $0 \leq n \leq 2N - 1$ . The result for  $n \geq 2N$  follows directly from (15) and [8, (14.9.3)] with  $\mu = 1/2 - N$ . This completes the proof.  $\blacksquare$

**Theorem 2.10.** Let  $n, m, N \in \mathbb{N}_0$ ,  $x \in (-1, 1)$ . Then  $S_{n,N}$  as defined in Lemma 2.9 fulfills the property of orthogonality

$$\mathcal{B}_N(S_{n,N}(x), S_{m,N}(x)) = \delta_{n,m} \begin{cases} h_{n,N}^I, & \text{if } n \leq 2N - 1, \\ h_{n,N}^{II}, & \text{if } n \geq 2N, \end{cases} \quad (42)$$

where

$$h_{n,N}^I = \frac{\left(\frac{1}{2}-N, 1-2N\right)_n}{2^{2N} \left(\frac{1}{2}\right)_N^2 n! \left(\frac{3}{2}-N\right)_n}, \quad h_{n,N}^{II} = \frac{2(n!)}{(2n-2N+1)(n-2N)!}. \quad (43)$$

The bilinear form  $\mathcal{B}_N$  is defined as follows [14, Theorem 4.3]

$$\mathcal{B}_N(\mathbf{f}, \mathbf{g}) := (\mathbf{f}, \mathbf{g})_{\pm 1} + (\mathbf{f}, \mathbf{g})_I = \begin{cases} (\mathbf{f}, \mathbf{g})_{\pm 1}, & \text{if } n \leq 2N - 1, \\ (\mathbf{f}, \mathbf{g})_I, & \text{if } n \geq 2N, \end{cases}, \quad (44)$$

where

$$(\mathbf{f}, \mathbf{g})_{\pm 1} := \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{2^{k-1}}{(2-2N)_k} \left[ \left( \frac{d^k}{dx^k} \mathbf{f}(x) \mathbf{g}(x) \right)_{x=1} + (-1)^k \left( \frac{d^k}{dx^k} \mathbf{f}(x) \mathbf{g}(x) \right)_{x=-1} \right], \quad (45)$$

$$(\mathbf{f}, \mathbf{g})_I := \int_{-1}^1 \mathbf{f}^{(2N)}(x) \mathbf{g}^{(2N)}(x) (1-x^2)^N dx. \quad (46)$$

**Proof.** Using the Sobolev orthogonality properties for Gegenbauer polynomials given in [1], [14], and then using Lemma 2.9 to convert the problem into one given in terms of Ferrers functions completes the proof for the Sobolev orthogonality. Let us now derive the values for the norms. The Sobolev orthogonality splits into two regimes: one where the degree of the polynomial  $n \leq 2N - 1$ , in which we get the norm  $h_{n,N}^I$  and another where  $n \geq 2N$  for which we get the norm  $h_{n,N}^{II}$ .

First we treat the case  $n \leq 2N - 1$ . By using the algebraic properties of the Gegenbauer polynomials (see [1]) and the fact that the integral vanishes because the Gegenbauer polynomials have degree less than  $2N$ , one obtains

$$h_{n,N}^I = \mathcal{B}(S_{n,N}(x), S_{n,N}(x)) = \frac{1}{2^{2N} \left(\frac{1}{2}\right)_N^2} \left( C_n^{\frac{1}{2}-N}(x), C_n^{\frac{1}{2}-N}(x) \right)_{\pm 1}.$$

By using (40) and the three term recurrence relation for the Gegenbauer polynomials [8, Table 18.9.1] we have

$$\begin{aligned} h_{n,N}^I &= \frac{1}{2^{2N} \left(\frac{1}{2}\right)_N^2} \left( C_n^{\frac{1}{2}-N}(x), \frac{2x(n-1)}{n+N} C_{n-1}^{\frac{1}{2}-N}(x) - \frac{n-2-N}{n+N} C_{n-2}^{\frac{1}{2}-N}(x) \right)_{\pm 1} \\ &= -\frac{2N-2n+1}{n2^{2N} \left(\frac{1}{2}\right)_N^2} \left( x C_n^{\frac{1}{2}-N}(x), C_{n-1}^{\frac{1}{2}-N}(x) \right)_{\pm 1} \\ &= -\frac{(2N-2n+1)(2N-n)}{n(2N-2n-1)2^{2N} \left(\frac{1}{2}\right)_N^2} \left( C_{n-1}^{\frac{1}{2}-N}(x), C_{n-1}^{\frac{1}{2}-N}(x) \right)_{\pm 1} \\ &= -\frac{(2N-2n+1)(2N-n)}{n(2N-2n-1)} h_{n-1,N}^I. \end{aligned}$$

Since  $h_{0,N}^I = 1/(2^{2N} \left(\frac{1}{2}\right)_N^2)$  one obtains the desired result by induction.

If  $n \geq 2N$ , then due to the following factorization identity [1, (10)]

$$C_n^{\frac{1}{2}-N}(x) = (-1)^N (1-x^2)^N C_{n-2N}^{\frac{1}{2}+N}(x),$$



using (13) and after some algebraic manipulations one demonstrates

$$\left( C_n^{\frac{1}{2}-N}(x), C_n^{\frac{1}{2}-N}(x) \right)_{\pm 1} = \left( (1-x^2)^{2N} C_{n-2N}^{\frac{1}{2}+N}(x), C_{n-2N}^{\frac{1}{2}+N}(x) \right)_{\pm 1} = 0.$$

Furthermore,

$$h_{n,N}^{II} = \int_{-1}^1 \left[ \frac{d^{2N}}{dx^{2N}} (1-x^2)^{\frac{N}{2}} P_{n-N}^{-N}(x) \right]^2 (1-x^2)^N dx,$$

and then taking into account the identity [4, Remark 14]

$$\frac{d^m}{dx^m} \frac{P_{\nu}^{\mu}(x)}{(1-x^2)^{\frac{\mu}{2}}} = \frac{(-1)^m P_{\nu}^{\mu+m}(x)}{(1-x^2)^{\frac{\mu+m}{2}}},$$

with  $m = 2N$ ,  $\nu = n - N$  and  $\mu = -N$  one obtains

$$h_{n,N}^{II} = \int_{-1}^1 P_{n-N}^N(x) P_{n-N}^N(x) dx = \frac{2(n!)}{(1+2n-2N)(n-2N)!},$$

by (32) with  $l = N$  and  $k = n - N$ . This completes the proof. ■

More details about the Gegenbauer/ultraspherical polynomials for non-classical parameters, their recurrence relations, and the Sobolev-type orthogonality can be found in [14].

### 3 Properties which follow from orthogonality and completeness

**Remark 3.1.** *The connection relation for Gegenbauer polynomials [8, (18.18.16)] produces the following result*

$$\begin{aligned} P_{n+\nu}^{-\nu}(x) &= \frac{2^{\nu-\mu} n! (1-x^2)^{\frac{\nu-\mu}{2}}}{\Gamma(2\nu+2+n)} \\ &\times \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (\mu+n-2k+\frac{1}{2}) \frac{(\nu-\mu)_k \Gamma(\nu+\frac{1}{2}+n-k) \Gamma(2\mu+n-2k+1)}{k!(n-2k)! \Gamma(\mu+\frac{3}{2}+n-k)} P_{n-2k+\mu}^{-\mu}(x), \end{aligned} \quad (47)$$

and the linearization formula for Gegenbauer polynomials [8, (18.18.22)] produces the following

$$\begin{aligned} P_{n+\lambda}^{-\lambda}(x) P_{m+\lambda}^{-\lambda}(x) &= \frac{2^{\lambda} \Gamma(\lambda+\frac{1}{2}) m! n! (1-x^2)^{\frac{\lambda}{2}}}{\sqrt{\pi} \Gamma(2\lambda+m+1) \Gamma(2\lambda+n+1)} \sum_{k=0}^m B_{k,m,n}^{\lambda+\frac{1}{2}} \frac{\Gamma(2\lambda+n+m-2k+1)}{(n+m-2k)!} P_{n+m-2k+\lambda}^{-\lambda}(x), \end{aligned} \quad (48)$$

where  $m, n \in \mathbb{N}_0$ ,  $n \geq m$ , and

$$B_{k,m,n}^{\lambda} := \frac{(m+n+\lambda-2k)(m+n-2k)! (\lambda)_k (\lambda)_{m-k} (\lambda)_{n-k} (2\lambda)_{m+n-k}}{(m+n+\lambda-k) k! (m-k)! (n-k)! (\lambda)_{m+n-k} (2\lambda)_{m+n-2k}}.$$

We leave the derivation of these simple consequences to the reader.

### 3.1 Some definite integrals and infinite series

One can use these orthogonality relations to compute some new definite integrals. Furthermore, using the derived closure relations one can convert certain definite integrals into infinite series expressions. In this subsection, we give some carefully chosen examples of these procedures.

The following result follows from a sample generating function for the Gegenbauer polynomials (50) and then re-expressing as a definite integral using orthogonality for  $P_{\nu}^{\mu}$ .

**Theorem 3.2.** *Let  $|x|, |t| < 1$ ,  $\alpha, \gamma \in \mathbb{C}$ ,  $k, \ell \in \mathbb{N}_0$ ,  $\alpha + k + \frac{1}{2} \neq 0$ ,  $\Re \alpha > -\ell - 1$ . Then*

$$\int_{-1}^1 (1+t^2-2xt)^{\frac{\alpha-\gamma+\ell}{2}} P_{\ell+\alpha-\gamma}^{-\ell-\alpha} \left( \frac{1-xt}{\sqrt{1+t^2-2xt}} \right) P_{k+\alpha}^{-\ell-\alpha}(x) dx = \frac{(-1)^{\ell}(\gamma-\ell)_k t^{\alpha+k}}{(1-\gamma)_{\ell} \Gamma(2\alpha+\ell+k+1)(\alpha+k+\frac{1}{2})}. \quad (49)$$

*Special care must be taken when  $\gamma \in \mathbb{Z}$ .*

**Proof.** Start with the generating function  $\lambda, \gamma \in \mathbb{C}$ , [8, (9.8.33)]

$$(1-xt)^{-\gamma} {}_2F_1 \left( \begin{matrix} \frac{\gamma}{2}, \frac{\gamma+1}{2} \\ \lambda + \frac{1}{2} \end{matrix}; \frac{t^2(x^2-1)}{(1-xt)^2} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(2\lambda)_n} C_n^{\lambda}(x) t^n, \quad (50)$$

and making the replacement  $n \mapsto k - \ell$ ,  $k \geq \ell \geq 0$  converts the sum over all  $k \geq \ell \in \mathbb{N}_0$ . Then convert the Gegenbauer polynomial to a Ferrers function of the first kind using (13). Multiply the resulting sum by  $(1-x^2)^{\frac{\alpha+\ell}{2}} P_{k'+\alpha}^{-\ell-\alpha}(x)$  and integrate both sides of the equation over  $(-1, 1)$  using the orthogonality relation (27) produces a definite integral of the Gauss hypergeometric function multiplied by a Ferrers function of the first kind. One can then convert the Gauss hypergeometric function to a Ferrers function of the first kind using (9) which completes the proof. ■

The following result follows from an expansion over Gegenbauer polynomials [3, Corollary 1] which generalizes many classical expansions including the generating function for Legendre polynomials, Heine's formula and Heine's reciprocal square root Fourier cosine series expansion [5, Figure 1].

**Lemma 3.3.** *Let  $\nu, \mu, x \in \mathbb{C}$  such that if  $z \in \mathbb{C} \setminus (-\infty, 1]$  lies on any ellipse with foci at  $\pm 1$  then  $x$  can lie on any point interior to that ellipse. Then*

$$\frac{(1-x^2)^{\frac{\mu}{2}} (z^2-1)^{\frac{\nu-\mu-1}{2}}}{(z-x)^{\nu}} = \sum_{n=0}^{\infty} \frac{(2\mu+2n+1)(\nu)_n \Gamma(2\mu+n+1)}{n!} P_{n+\mu}^{-\mu}(x) Q_{n+\mu}^{\nu-\mu-1}(z). \quad (51)$$

**Proof.** Start with [3, Corollary 1] and convert the Gegenbauer polynomial to a Ferrers function of the first kind using (15), then set  $\mu \mapsto \mu + \frac{1}{2}$ . This completes the proof. ■

**Theorem 3.4.** *Let  $n, l \in \mathbb{N}_0$ ,  $n \geq l$ ,  $\nu, \mu \in \mathbb{C}$ , such that  $z \in \mathbb{C} \setminus (-\infty, 1]$ . Then*

$$\int_{-1}^1 \frac{P_{n+\mu}^{-\mu}(x)}{(z-x)^{\nu}} (1-x^2)^{\frac{\mu}{2}} dx = \frac{2(\nu)_n Q_{n+\mu}^{\nu-\mu-1}(z)}{(z^2-1)^{\frac{\nu-\mu-1}{2}}}. \quad (52)$$

**Proof.** Starting with (51), multiplying both sides of the equation by  $P_{n+\mu}^{-\mu}(x)$  and integrating with respect to  $x$  over  $(-1, 1)$  using the orthogonality relation (27) completes the proof. ■

The following result follows from an integral derived in [2] by Askey and Razban for the Jacobi polynomials, but then we specialize to Gegenbauer polynomials and then re-express using  $P_{\nu}^{\mu}$ .

**Theorem 3.5.** *Let  $n \in \mathbb{N}_0$ ,  $\lambda, \gamma \in \mathbb{C}$ . Then*

$$\int_{-1}^1 (1-x)^{\frac{\lambda}{2}-\gamma}(1+x)^{\frac{\lambda}{2}} P_{n+\lambda}^{-\lambda}(x) dx = \frac{2^{\lambda-\gamma+1} \Gamma(\lambda-\gamma+1) \Gamma(\gamma+n)}{\Gamma(\gamma) \Gamma(2\lambda-\gamma+n+2)}. \quad (53)$$

**Proof.** Start with [2] (see also [10, (16.4.2)])

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x) (1-x)^{\alpha-\gamma}(1+x)^{\beta} dx = \frac{2^{\alpha+\beta-\gamma+1} \Gamma(\alpha-\gamma+1) \Gamma(\beta+n+1) \Gamma(\gamma+n)}{n! \Gamma(\gamma) \Gamma(\alpha+\beta-\gamma+n+2)},$$

then replace  $\alpha, \beta \mapsto \lambda - \frac{1}{2}$  and using (5), then (15) produces the following integral over Gegenbauer polynomials

$$\int_{-1}^1 (1-x)^{\lambda-\gamma-\frac{1}{2}}(1+x)^{\lambda-\frac{1}{2}} C_n^{\lambda}(x) dx = \frac{\sqrt{\pi} \Gamma(\lambda-\gamma+\frac{1}{2}) \Gamma(2\lambda+n) \Gamma(\gamma+n)}{2^{\gamma-1} n! \Gamma(\lambda) \Gamma(\gamma) \Gamma(2\lambda-\gamma+n+1)}, \quad (54)$$

using (15) and replacing  $\lambda \mapsto \lambda + \frac{1}{2}$  completes the proof.  $\blacksquare$

**Remark 3.6.** *The above result is equivalent to the following expansion*

$$\sum_{n=0}^{\infty} \frac{(\lambda+n)(\gamma)_n}{(2\lambda-\gamma+1)_n} C_n^{\lambda}(x) = \frac{\sqrt{\pi} \Gamma(2\lambda-\gamma+1)(1-x)^{-\gamma}}{2^{2\lambda-\gamma} \gamma(\lambda) \Gamma(\lambda-\gamma+\frac{1}{2})}, \quad (55)$$

which we recently proved in [5, Corollary 5.12].

### 3.2 Christoffel-Darboux formulas

The Christoffel-Darboux formula for orthogonal Ferrers functions is given in the following theorem which gives the Christoffel-Darboux formula and its confluent form for the Ferrers function of the first kind.

**Theorem 3.7.** *Let  $x, y \in (-1, 1)$ ,  $x \neq y$ ,  $\mu \in \mathbb{C}$ ,  $n \in \mathbb{N}_0$ . Then, the Christoffel-Darboux formula and its confluent form for  $P_{\nu}^{\mu}$  are given by*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(2\mu+2k+1) \Gamma(2\mu+k+1)}{k!} P_{k+\mu}^{-\mu}(x) P_{k+\mu}^{-\mu}(y) \\ &= \frac{\Gamma(2\mu+n+1)}{(n-1)!(x-y)} \left( P_{n+\mu}^{-\mu}(x) P_{n+\mu-1}^{-\mu}(y) - P_{n+\mu-1}^{-\mu}(x) P_{n+\mu}^{-\mu}(y) \right) \end{aligned} \quad (56)$$

and

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(2\mu+2k+1) \Gamma(2\mu+k+1)}{k!} \left( P_{k+\mu}^{-\mu}(x) \right)^2 = \frac{\Gamma(2\mu+n+1)}{(n-1)!(1-x^2)} \left( (n+2\mu) \left( P_{n+\mu}^{-\mu}(x) \right)^2 \right. \\ & \quad \left. + n \left( P_{n+\mu-1}^{-\mu}(x) \right)^2 - 2x(n+\mu) P_{n+\mu-1}^{-\mu}(x) P_{n+\mu}^{-\mu}(x) \right). \end{aligned} \quad (57)$$

**Proof.** Start with the Christoffel-Darboux summation formula for the Gegenbauer polynomials [16, (3.10)]

$$\sum_{k=0}^{n-1} \frac{k!(k+\lambda)}{\Gamma(2\lambda+k)} C_k^{\lambda}(x) C_k^{\lambda}(y) = \frac{n! \left( C_n^{\lambda}(x) C_{n-1}^{\lambda}(y) - C_{n-1}^{\lambda}(x) C_n^{\lambda}(y) \right)}{2\Gamma(2\lambda-1+n)(x-y)}. \quad (58)$$

The confluent form of the Christoffel-Darboux summation formula follows from the first formula using [8, (18.2.13)] and recurrence relations for the Gegenbauer polynomials [12, (9.8.21)], and is given by

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{k!(k+\lambda)}{\Gamma(2\lambda+k)} (C_k^\lambda(x))^2 \\ &= \frac{n! \left( n(C_n^\lambda(x))^2 + (2\lambda+n-1)(C_{n-1}^\lambda(x))^2 - x(2\lambda+2n-1)C_n^\lambda(x)C_{n-1}^\lambda(x) \right)}{2(1-x^2)\Gamma(2\lambda-1+n)}. \end{aligned} \quad (59)$$

The conversion to the Ferrers functions of the first kind is accomplished using (15). This completes the proof.  $\blacksquare$

Using (12), the above Christoffel-Darboux formulas for  $P_\nu^\mu$  can be expressed in terms of  $Q_\nu^\mu$ .

**Corollary 3.8.** *Let  $n \in \mathbb{N}_0$ ,  $x, y \in (-1, 1)$ ,  $x \neq y$ . Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(n+k)k!}{(2n+k-1)!} Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(y) \\ &= \frac{n!}{2(3n-2)!(x-y)} \left( Q_{2n-\frac{1}{2}}^{n-\frac{1}{2}}(x) Q_{2n-\frac{3}{2}}^{n-\frac{1}{2}}(y) - Q_{2n-\frac{3}{2}}^{n-\frac{1}{2}}(x) Q_{2n-\frac{1}{2}}^{n-\frac{1}{2}}(y) \right) \end{aligned} \quad (60)$$

and

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(n+k)k!}{(2n+k-1)!} \left( Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) \right)^2 = \frac{n!}{2(3n-2)!(1-x^2)} \left( n \left( Q_{2n-\frac{1}{2}}^{n-\frac{1}{2}}(x) \right)^2 \right. \\ & \quad \left. + (3n-1) \left( Q_{2n-\frac{3}{2}}^{n-\frac{1}{2}}(x) \right)^2 - x(4n-1) Q_{2n-\frac{3}{2}}^{n-\frac{1}{2}}(x) Q_{2n-\frac{1}{2}}^{n-\frac{1}{2}}(x) \right). \end{aligned} \quad (61)$$

**Proof.** Let  $\mu = n - \frac{1}{2}$  in (56), (57), respectively using (12) completes the proof.  $\blacksquare$

### 3.3 Poisson kernels

The Poisson kernel  $K_t(x, y)$  of an orthogonal polynomial  $p_n(x; \mathbf{a})$ , where  $\mathbf{a}$  is a set of parameters that the orthogonal polynomial depends upon is given by

$$K_t(x, y; \mathbf{a}) = \sum_{n=0}^{\infty} \frac{t^n}{h_n} p_n(x; \mathbf{a}) p_n(y; \mathbf{a}), \quad (62)$$

where  $h_n$  is defined in (26). In this subsection we start with the Poisson kernel of Gegenbauer polynomials and some companion identities to derive addition theorems for the Ferrers function of the first kind. In fact, in Remark 3.10 below we identify the connection of the Poisson kernel for Gegenbauer polynomials with the *second* addition theorem for spherical harmonics which sums over all meridional quantum numbers  $n$  for a fixed azimuthal quantum number  $m$  given in [7, (2.4)]. This corresponds to a toroidal harmonic and is constant on toroids of revolution.

**Theorem 3.9.** *Let  $t, \lambda \in \mathbb{C}$ ,  $|t| < 1$ ,  $x, y \in (-1, 1)$ ,  $x = \cos \theta$ ,  $y = \cos \phi$ . Then*

$$\sum_{n=0}^{\infty} \frac{(2\lambda+1)_n t^n}{n!} P_{n+\lambda}^{-\lambda}(x) P_{n+\lambda}^{-\lambda}(y) = \frac{Q_{\lambda-\frac{1}{2}}(\chi)}{\pi \Gamma(2\lambda+1) t^{\lambda+\frac{1}{2}} \sqrt{\sin \theta \sin \phi}}, \quad (63)$$

where

$$\chi := \frac{1+t^2-2t \cos \theta \cos \phi}{2 \sin \theta \sin \phi}. \quad (64)$$

**Proof.** Start with the Poisson kernel for the Gegenbauer polynomials [13, (19)]

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n!t^n}{(2\lambda)_n} C_n^\lambda(x) C_n^\lambda(y) &= \frac{1}{(1+t^2-2t\cos\theta\cos\phi)^\lambda} {}_2F_1\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{\chi^2}\right) \\ &= \frac{\Gamma(\lambda + \frac{1}{2}) Q_{\lambda-1}(\chi)}{\sqrt{\pi} \Gamma(\lambda) t^\lambda (1-x^2)^{\frac{\lambda}{2}} (1-y^2)^{\frac{\lambda}{2}}}, \end{aligned} \quad (65)$$

then replacing  $\lambda \mapsto \lambda + \frac{1}{2}$ , and using (15) completes the proof.  $\blacksquare$

**Remark 3.10.** Note that for  $\lambda = m$ , and setting  $n \mapsto n + m$ , Theorem 3.9 specializes to the second addition theorem for spherical harmonics presented in [7, (2.4)]

$$\sum_{n=|m|}^{\infty} t^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\phi) = \frac{Q_{m-\frac{1}{2}}(\chi)}{\pi \sqrt{t \sin\theta \sin\phi}}, \quad (66)$$

after letting  $t = r_{<}/r_{>}$ , where  $r_{\leq} := \frac{\min\{r, r'\}}{\max\{r, r'\}}$  with  $r, r' \in [0, \infty)$ .

**Theorem 3.11.** Let  $t, \lambda \in \mathbb{C}$ ,  $|t| < 1$ ,  $x, y \in (-1, 1)$ ,  $x = \cos\theta$ ,  $y = \cos\phi$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda+n+\frac{1}{2})(2\lambda+1)_n t^n}{n!} P_{n+\lambda}^{-\lambda}(x) P_{n+\lambda}^{-\lambda}(y) \\ = -\frac{1-t^2}{2\pi \Gamma(2\lambda+1) t^{\lambda+\frac{3}{2}} (1-x^2)^{\frac{3}{4}} (1-y^2)^{\frac{3}{4}}} \frac{Q_{\lambda-\frac{1}{2}}^1(\chi)}{\sqrt{\chi^2-1}}. \end{aligned} \quad (67)$$

**Proof.** Start with the companion Poisson kernel for Gegenbauer polynomials [13, (18)]

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\lambda+n}{\lambda} \frac{n!t^n}{(2\lambda)_n} C_n^\lambda(x) C_n^\lambda(y) &= \frac{1-t^2}{(1+t^2-2t\cos\theta\cos\phi)^{\lambda+1}} {}_2F_1\left(\frac{\lambda+1}{2}, \frac{\lambda+2}{2}; \frac{1}{\chi^2}\right) \\ &= -\frac{\Gamma(\lambda + \frac{1}{2})(1-t^2)}{\sqrt{\pi} \Gamma(\lambda+1) t^{\lambda+1} (1-x^2)^{\frac{\lambda+1}{2}} (1-y^2)^{\frac{\lambda+1}{2}}} \frac{Q_{\lambda-1}^1(\chi)}{\sqrt{\chi^2-1}}, \end{aligned} \quad (68)$$

then replacing  $\lambda \mapsto \lambda + \frac{1}{2}$ , and using (15) completes the proof.  $\blacksquare$

**Theorem 3.12.** Let  $t, \lambda \in \mathbb{C}$ ,  $|t| < 1$ ,  $x, y \in (-1, 1)$ ,  $x = \cos\theta$ ,  $y = \cos\phi$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda+n+\frac{1}{2})(\lambda+n+\frac{3}{2})(2\lambda+1)_n t^n}{n!} P_{n+\lambda}^{-\lambda}(x) P_{n+\lambda}^{-\lambda}(y) \\ = \frac{1}{\pi t^{\lambda-\frac{1}{2}} \Gamma(2\lambda+1) (1-x^2)^{\frac{3}{4}} (1-y^2)^{\frac{3}{4}}} \left( \frac{Q_{\lambda-\frac{1}{2}}^1(\chi)}{\sqrt{\chi^2-1}} + \frac{(1-t^2)^2 Q_{\lambda-\frac{1}{2}}^2(\chi)}{4t^3 \sqrt{1-x^2} \sqrt{1-y^2} (\chi^2-1)} \right). \end{aligned} \quad (69)$$

**Proof.** Start with Theorem 3.11, and put all powers of  $t$  on the left-hand side of the equation and differentiate with respect to  $t$  using

$$\frac{d}{dt} \left( \frac{t^{\lambda+n+\frac{3}{2}}}{1-t^2} \right) = \frac{2t^{\lambda+n+\frac{5}{2}}}{(1-t^2)^2} + \frac{(\lambda+n+\frac{3}{2})t^{\lambda+n+\frac{1}{2}}}{1-t^2},$$

$$\frac{\partial \chi}{\partial t} = \frac{-(1-t^2)}{2t^2 \sqrt{1-x^2} \sqrt{1-y^2}},$$

and the following derivative formula for the associated Legendre function of the second kind (cf. [4, Remark 4])  $d/d\chi Q_{\lambda-\frac{1}{2}}^m(\chi)/(1-\chi^2)^{\frac{m}{2}} = Q_{\lambda-\frac{1}{2}}^{m+1}(\chi)/(\chi^2-1)^{\frac{m+1}{2}}$ , for  $m \in \mathbb{N}_0$ . Then applying Theorem 3.11 again and the recurrence relation for  $Q_\nu^\mu$  [8, (14.10.7)] completes the proof.  $\blacksquare$

**Corollary 3.13.** *Let  $t, \lambda \in \mathbb{C}$ ,  $|t| < 1$ ,  $x, y \in (-1, 1)$ ,  $x = \cos \theta$ ,  $y = \cos \phi$ . Then*

$$\begin{aligned} & \sum_{n=0}^{\infty} (\lambda + n)(\lambda + n + 1) \frac{n!t^n}{(2\lambda)_n} C_n^\lambda(x) C_n^\lambda(y) \\ &= \frac{\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi} t^{\lambda-1} \Gamma(\lambda) (1-x^2)^{\frac{\lambda+1}{2}} (1-y^2)^{\frac{\lambda+1}{2}}} \left( \frac{Q_{\lambda-1}^1(\chi)}{\sqrt{\chi^2-1}} + \frac{(1-t^2)^2 Q_{\lambda-1}^2(\chi)}{4t^3 \sqrt{1-x^2} \sqrt{1-y^2} (\chi^2-1)} \right). \end{aligned} \quad (70)$$

**Proof.** Using (15) in Theorem 3.12, completes the proof.  $\blacksquare$

**Theorem 3.14.** *Let  $t, \lambda \in \mathbb{C}$ ,  $|t| < 1$ ,  $x \in (-1, 1)$ ,  $x = \cos \theta$ . Then, one has the following algebraic generating functions for the Gegenbauer polynomials:*

$$\sum_{n=0}^{\infty} t^n C_n^\lambda(x) = \frac{1}{(1+t^2-2tx)^\lambda}, \quad (71)$$

$$\sum_{n=0}^{\infty} (\lambda + n) t^n C_n^\lambda(x) = \frac{\lambda(1-t^2)}{(1+t^2-2tx)^{\lambda+1}}, \quad (72)$$

$$\sum_{n=0}^{\infty} (\lambda + n)(\lambda + n + 1) t^n C_n^\lambda(x) = \frac{\lambda(\lambda + 1)(1-t^2)^2}{(1+t^2-2tx)^{\lambda+2}} - \frac{2\lambda t^2}{(1+t^2-2tx)^{\lambda+1}}. \quad (73)$$

**Proof.** Consider the limit as  $y \rightarrow 1^-$  as follows. Let  $y = 1 - \frac{1}{2}\epsilon^2$  with  $\epsilon > 0$ , such that  $\epsilon \ll 1$ . Then  $\sqrt{1-y^2} \sim \epsilon$  and  $\chi \sim (1+t^2-2tx)/(2t\sqrt{1-x^2}\epsilon)$ , as  $\epsilon \rightarrow 0^+$ . Note that [8, Table 18.6.1]  $C_n^\lambda(1) = (2\lambda)_n/n!$ . Applying this and the above estimates in (65), (68) and Theorem 3.13, then taking the limit as  $\epsilon \rightarrow 0^+$ , completes the proof.  $\blacksquare$

**Remark 3.15.** *A limiting case of the above theorem produces the following results for Ferrers functions of the first kind.*

$$\sum_{n=0}^{\infty} \frac{t^n (2\lambda + 1)_n}{n!} P_{n+\lambda}^{-\lambda}(x) = \frac{(1-x^2)^{\frac{\lambda}{2}}}{2^\lambda \Gamma(\lambda + 1) (1+t^2-2tx)^{\lambda+\frac{1}{2}}}, \quad (74)$$

$$\sum_{n=0}^{\infty} (\lambda + n + \frac{1}{2}) \frac{t^n (2\lambda + 1)_n}{n!} P_{n+\lambda}^{-\lambda}(x) = \frac{(\lambda + \frac{1}{2})(1-t^2)(1-x^2)^{\frac{\lambda}{2}}}{2^\lambda \Gamma(\lambda + 1) (1+t^2-2tx)^{\lambda+\frac{3}{2}}}, \quad (75)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} (\lambda + n + \frac{1}{2})(\lambda + n + \frac{3}{2}) \frac{t^n (2\lambda + 1)_n}{n!} P_{n+\lambda}^{-\lambda}(x) \\ &= \frac{(1-x^2)^{\frac{\lambda}{2}} (\lambda + \frac{1}{2})}{2^\lambda \Gamma(\lambda + 1)} \left( \frac{(\lambda + \frac{3}{2})(1-t^2)^2}{(1+t^2-2tx)^{\lambda+\frac{5}{2}}} - \frac{2t^2}{(1+t^2-2tx)^{\lambda+\frac{3}{2}}} \right). \end{aligned} \quad (76)$$

### 3.4 Closure relations

*Closure relations* are infinite series representations of the Dirac delta distribution. They are obtained from a complete set of orthogonal functions. See [11, Theorem 2.1] for a proof of the following proposition.

**Proposition 3.16.** *Let  $x, y \in [a, b]$  and the Poisson kernel is non-negative. Then, one has the following closure relation:*

$$\sum_{n=0}^{\infty} \frac{w(x)}{h_n} \psi_n(x) \psi_n(y) = \delta(x-y), \quad (77)$$

where both the left- and right-hand sides should be treated as distributions.

The following is the closure relation for  $P_\nu^\mu$ .

**Theorem 3.17.** *Let  $x, y \in (-1, 1)$ ,  $\mu \in \mathbb{C}$ ,  $2\mu + 1 \notin -\mathbb{N}_0$ . Then*

$$\sum_{n=0}^{\infty} \frac{(\mu + n + \frac{1}{2})\Gamma(2\mu + n + 1)}{n!} P_{n+\mu}^{-\mu}(x) P_{n+\mu}^{-\mu}(y) = \delta(x - y). \quad (78)$$

**Proof.** Using the orthogonality relation for Gegenbauer polynomials (28) in order to obtain  $h_n$  with (77) produces

$$\sum_{n=0}^{\infty} \frac{(\mu + n)n!}{\Gamma(2\mu + n)} C_n^\mu(x) C_n^\mu(y) = \frac{\pi\delta(x - y)}{(1 - x^2)^{\mu - \frac{1}{2}} 2^{2\mu - 1} \Gamma^2(\mu)}. \quad (79)$$

Applying (15) twice completes the proof. ■

**Remark 3.18.** *Note that in the limit as  $n \rightarrow \infty$  the Christoffel-Darboux formula (56) becomes the closure relation (78). Furthermore, the Poisson kernel can be obtained from the Christoffel-Darboux sum with each coefficient multiplied by some  $t^k$ , with  $|t| < 1$ . This interrelation and the connection with universality for orthogonal polynomials are quite intriguing (see the excellent review article by Barry Simon entitled “The Christoffel-Darboux Kernel” [15, pp. 295–335]).*

**Corollary 3.19.** *Let  $x \in (-1, 1)$ ,  $\lambda, \gamma \in \mathbb{C}$  such that  $2\lambda - \gamma + 2 \notin -\mathbb{N}_0$ . Then*

$$\sum_{n=0}^{\infty} \frac{(\lambda + n + \frac{1}{2})(\gamma)_n (2\lambda + 1)_n}{n!(2\lambda - \gamma + 2)_n} P_{n+\lambda}^{-\lambda}(x) = \frac{\Gamma(2\lambda - \gamma + 2)(1 - x)^{\frac{\lambda}{2} - \gamma} (1 + x)^{\frac{\lambda}{2}}}{2^{\lambda - \gamma + 1} \Gamma(2\lambda + 1) \Gamma(\lambda - \gamma + 1)}. \quad (80)$$

**Proof.** Applying the closure relation (78) to (53), multiplying by the necessary factors and integrating over  $(-1, 1)$  completes the proof. ■

**Remark 3.20.** *It is interesting to see that using orthogonality for Jacobi polynomials [8, Table 18.3.1], the closure relation for Jacobi polynomials is given by*

$$\sum_{n=0}^{\infty} \frac{(2n + \alpha + \beta + 1)\Gamma(\alpha + \beta + n + 1)n!}{\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)} P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) = \frac{2^{\alpha + \beta + 1} \delta(x - y)}{(1 - x)^\alpha (1 + x)^\beta}. \quad (81)$$

**Remark 3.21.** *It should be noted that if one takes  $\mu = 0$  in (78) then one obtains the closure relation for Legendre polynomials (18) [8, (1.17.22)]*

$$\sum_{n=0}^{\infty} (n + \frac{1}{2}) P_n(x) P_n(y) = \delta(x - y). \quad (82)$$

*On the other hand, using  $\mu = \pm 1/2$  in (78) produces the following closure relations for the Chebyshev polynomials of the first and second kinds [8, Table 18.3.1], namely:*

$$\sum_{n=0}^{\infty} T_n(x) T_n(y) = \frac{\pi}{2} (1 - x^2)^{\frac{1}{4}} (1 - y^2)^{\frac{1}{4}} \delta(x - y), \quad (83)$$

$$\sum_{n=0}^{\infty} U_n(x) U_n(y) = \frac{\pi\delta(x - y)}{2(1 - x^2)^{\frac{1}{4}} (1 - y^2)^{\frac{1}{4}}}. \quad (84)$$

Using (34) we can obtain a closure relation for  $Q_\nu^\mu$ .

**Corollary 3.22.** *Let  $n \in \mathbb{N}_0$ ,  $x, y \in (-1, 1)$ . Then*

$$\sum_{k=0}^{\infty} \frac{(n+k)k!}{(2n+k-1)!} Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(x) Q_{k+n-\frac{1}{2}}^{n-\frac{1}{2}}(y) = \frac{\pi^2}{4} \delta(x-y). \quad (85)$$

**Proof.** From (34) we obtain  $h_n = \pi^2(2n+k-1)!/(4(n+k)k!)$ , then using (77) with  $w(x) = 1$  completes the proof. ■

The  $n = 0, 1$  specializations of (85) correspond with the closure relation for the Chebyshev polynomials of the first and second kinds (83), (84).

**Corollary 3.23.** *Let  $x, y \in (-1, 1)$ ,  $x = \cos \theta$ ,  $y = \cos \phi$ . Then*

$$\sum_{k=0}^{\infty} k^2 Q_{k-\frac{1}{2}}^{-\frac{1}{2}}(x) Q_{k-\frac{1}{2}}^{-\frac{1}{2}}(y) = \frac{\pi}{2\sqrt{\sin \theta \sin \phi}} \sum_{k=0}^{\infty} \cos(k\theta) \cos(k\phi) = \frac{\pi^2}{4} \delta(x-y), \quad (86)$$

$$\sum_{k=0}^{\infty} Q_{k+\frac{1}{2}}^{\frac{1}{2}}(x) Q_{k+\frac{1}{2}}^{\frac{1}{2}}(y) = \frac{\pi}{2\sqrt{\sin \theta \sin \phi}} \sum_{k=1}^{\infty} \sin(k\theta) \sin(k\phi) = \frac{\pi^2}{4} \delta(x-y). \quad (87)$$

**Proof.** Start with (85) with  $n = 0, 1$ , and using (10), (11), completes the proof. ■

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