

LOCALIZATION OF ZEROS OF POLAR POLYNOMIALS ON THE UNIT DISK

R. S. COSTAS-SANTOS AND A. REHOUMA

ABSTRACT. In the paper we derive useful result about the zeros of the k -polar polynomials on the unit circle, in particular we obtain a circular region containing all the zeros of these polynomials. Some examples are presented.

1. Introduction

Finding the roots of polynomials is a problem of interest in both mathematics and in areas of application such as physical systems, which can be reduced to solving certain equation. There are very interesting geometric relationships between the roots of a polynomial $f_n(z)$ and those of $f'_n(z)$. The most important result is the following.

Theorem 1.1 (The Gauß-Lucas theorem [12]). *Let $f_n(z) \in \mathbb{C}[z]$ be a polynomial of degree at least one. All zeros of $f'_n(z)$ lie in the convex hull of the zeros of $f_n(z)$.*

The location of zeros, or critical points, of polynomials has many physical and geometrical interpretations. For example, F. Gauß in 1816 showed that the roots of $f'_n(z)$ are the positions of equilibrium in the field of force due to equal particles situated at each root of $f_n(z)$, if each particle repels with a force equal to the inverse distance being the inverse distance law.

Many results exist concerning the location of the zeros of a polynomial of a complex variable as a function of the coefficients of the polynomial. One is the well-known Eneström-Kakeya theorem, [19]. Another one, useful to obtain more precise about the zeros of a polynomial, was obtained by J. H. Grace.

Theorem 1.2 (The Grace's theorem [9]). *Let $a(z)$ and $b(z)$ the following polynomials:*

$$a(z) = \sum_{\ell=0}^n a_{\ell} \binom{n}{\ell} z^{\ell}, \quad b(z) = \sum_{\ell=0}^n b_{\ell} \binom{n}{\ell} z^{\ell}.$$

If the zeros of both polynomials lie in the unit disk, then the zeros of the "composition" of the two

$$c(z) = \sum_{\ell=0}^n a_{\ell} b_{\ell} \binom{n}{\ell} z^{\ell},$$

also lie in the unit disk.

2020 Mathematics Subject Classification. 42C05; 33C45.

Key words and phrases. orthogonal polynomials; polar polynomials; Laurent series; unit circle; Location of zeros.

The zeros of orthogonal polynomials has been an intensively studied subject since the beginning of the 20th century, and several breakthroughs have been made in the recent years. In particular, Barry Simon proved in [19] that if is μ a finite positive measure defined on the Borelian σ -algebra of \mathbb{C} , μ is absolutely continuous with respect to the Lebesgue measure $d\theta/(2\pi)$ on $[-\pi, \pi]$; and $(L_n(z))$ is the system of monic orthogonal polynomials with respect to μ , i.e.

$$(1) \quad \int_{-\pi}^{\pi} L_n(z) z^{-j} d\mu(\theta) = 0, \quad j = 0, 1, \dots, n-1,$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} L_n(z) \overline{L_n(z)} d\mu(\theta) = \|L_n\|^2 \neq 0, \quad n = 0, 1, \dots,$$

where $z = \exp(i\theta)$ and $d\mu(\theta) = \rho(\theta)d\theta$ with $\rho \in L^1([-\pi, \pi], d\theta)$, is a measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Then, all the zeros of $L_n(z)$ are contained in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$

In this contribution we are going to introduce monic k -polar polynomials.

Definition 1.3. Let μ be a finite measure defined on the Borelian σ -algebra of \mathbb{C} such that it contains an infinite number of points and let $(L_n(z))$ the system of monic orthogonal polynomials with respect to μ . Let ξ be a fixed complex number. Let k be a positive integer, the k -polar polynomial related to μ , which will be denoted by $Q_{n;k}(z; \xi)$, is the polynomial solution of degree n of the k -th order linear differential equation

$$\frac{d^k}{dz^k} (z - \xi)^k P(z) = (n+1) \cdots (n+k) L_n(z).$$

Remark 1.4. Notice that, by construction, $Q_{n;0}(z) = L_n(z)$ for all n .

In the last years some attention has been paid to the so called polar orthogonal polynomials. Fundora, Pijeira, and Urbina [6] have studied 1-polar orthogonal polynomial sequences associated with a measure supported on the segment. A similar study has been done by Pijeira, Bello, and Urbina [18], in the case of 1-polar Legendre polynomials.

The main result in this paper is to study the zero location of k -polar orthogonal polynomials on the unit circle, in short OPUC, with respect to a generic measure μ .

The structure of this work is the following: in Section 2 we present some preliminaries and basic results we need to obtain the main result. In Section 3 we state the main result of this work about the location of the zeros of polar polynomials. In Section 4 we present three interesting examples of k -polar orthogonal polynomials on the unit circle. Since extensive calculations indicate that these polynomials often have complex zeros and there exists a ring-shaped region containing all the zeros of polar orthogonal we also present in this Section numerical calculations to see if Sendov's conjecture [14, p. 267] holds true or not for such examples.

2. Preliminaries

Given a complex number $z_0 \in \mathbb{C}$ and a radius $r > 0$, we define the open disk

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\},$$

1 the closed disk

$$2 \quad \overline{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\},$$

3 and the circle

$$4 \quad \partial D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}.$$

5 Let μ be a finite positive Borel measure that is absolutely continuous with respect to the Lebesgue
6 measure $d\theta/(2\pi)$ on $[-\pi, \pi]$. Let $(L_n(z))$ be the system of monic polynomials orthogonal with respect
7 to μ . It is known that the following mutually equivalent recursions relations hold for these polynomials:

$$8 \quad L_n(z) = zL_{n-1}(z) + L_n(0)L_{n-1}^*(z),$$

$$9 \quad L_n^*(z) = L_{n-1}^*(z) + \overline{zL_n(0)}L_{n-1}(z),$$

12 where

$$13 \quad (2) \quad L_n^*(z) = z^n \overline{L_n(1/\bar{z})} = 1 + z \sum_{\ell=0}^{n-1} \overline{L_{\ell+1}(0)} L_\ell(z), \quad z \neq 0.$$

16 Moreover, for $|z| = 1$, we have

$$17 \quad (3) \quad \left| \frac{L_{n+1}^*(z)}{L_n^*(z)} - 1 \right| = \left| \frac{L_{n+1}(z)}{L_n(z)} - z \right| = |L_{n+1}(0)|, \quad n = 0, 1, \dots$$

19 for more details about these former identities see [11, 17, 19, 20, 21].

20 Along this work we are going to deal with the zeros of polynomials with complex coefficients,
21 therefore it is convenient to state the following results.

23 **Theorem 2.1** (Szegő's theorem [20], [2]). *Let $a(z)$, $b(z)$ and $c(z)$ the polynomials defined in Theorem
24 1.2. If all the zeros of $a(z)$ lie in a closed disk \overline{D} and $\lambda_1, \dots, \lambda_n$ are the zeros of $b(z)$, then every zero of
25 $c(z)$ has the form $\lambda_\ell \gamma_\ell$, where $\gamma_\ell \in \overline{D}$.*

26 **Lemma 2.2** (Cauchy's. Theorem (27,2) in [13]). *If $P(z) = a_n z^n + \dots + a_1 z + a_0$ is a complex polynomial
27 of degree at least one, then all the zeros of P lie in a closed circle*

$$28 \quad |z| \leq 1 + A,$$

30 where $A = \max\{|a_0|, \dots, |a_{n-1}|\}/|a_n|$.

32 **Lemma 2.3** (Datt and Govil [3]). *If $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ is a complex polynomial of degree
33 at least one, then all the zeros of P lie in a ring shaped region*

$$34 \quad \frac{|a_0|}{2(1+B)^{n-1}(1+nB)} \leq |z| \leq 1 + \lambda_0 B,$$

36 where $B = \max\{|a_0|, \dots, |a_{n-1}|\}$, and λ_0 is the unique root of the equation $(x-1)(1+Bx)^n + 1 = 0$ in
37 the interval $[0, 1]$.

39 For more information about inequalities that satisfy the zeros of complex polynomials, read the
40 survey [16] and the references therein.

41 Recall that the convex hull of a set G is the smallest convex set containing G , that is, the intersection
42 of all closed containing G .

1 Sendov's conjecture [15, p. 267] asserts that if a polynomial $f(z)$ of degree $n \geq 2$ has all of its zeros
2 in \mathbb{D} , then for each such zero z_0 there is a zero of the derivative $f'(z)$ in the $\overline{D(z_0, 1)}$.

3 We are going to handle with k -polar polynomials. In the $k = 1$ case it is straightforward to obtain

$$4 \quad (4) \quad (n+1) \int_{\xi}^z L_n(t) dt = (z - \xi) Q_{n;1}(z; \xi),$$

6 therefore it is logical to call $Q_{n;1}(z; \xi)$ the n -th first order polar polynomial of $L_n(z)$ (see [6, 18]). As a
7 consequence of (4) we obtain

$$8 \quad (5) \quad (n+1)L_n(z) = Q_{n;1}(z; \xi) + (z - \xi)Q'_{n;1}(z; \xi).$$

10 **Remark 2.4.** Observe that, by construction, $Q_{n;1}(z; \xi)$ is a monic polynomial of degree n and the pole
11 of this polynomial is not irregular. In fact,

$$12 \quad \lim_{z \rightarrow \xi} Q_{n;1}(z; \xi) = \lim_{z \rightarrow \xi} \frac{(n+1) \int_{\xi}^z L_n(t) dt}{z - \xi} = (n+1)L_n(\xi).$$

15 Analogous calculations can be done in order to see that the k -polar monic polynomial $Q_{n;k}(z; \xi)$ has
16 degree n and the pole of this polynomial is not irregular.

18 3. Localization of zeros of polar polynomials

20 Our next propose is to prove that all the zeros of the k -polar monic polynomial $Q_{n;k}(z; \xi)$ are contained
21 in a disk which radius is independent of n . First, let us express the polynomials $L_n(z)$ and $Q_{n;k}(z; \xi)$ in
22 terms of power of $z - \xi$, that is

$$23 \quad (6) \quad L_n(z) = \sum_{\ell=0}^n a_{n,\ell} (z - \xi)^\ell, \quad Q_{n;k}(z) = \sum_{\ell=0}^n b_{n,\ell;k} (z - \xi)^\ell, \quad k = 1, 2, \dots,$$

25 where $a_{n,n} = b_{n,n;k} = 1$ for all $k = 1, 2, \dots$

27 **Lemma 3.1.** Let k be a positive integer. Let $\xi \in \mathbb{C}$. The coefficients of $L_n(z)$ and $Q_{n;k}(z; \xi)$ are fulfill
28 the following relation:

$$29 \quad (7) \quad b_{n,\ell;k} = \frac{(n+k) \cdots (n+1)}{(\ell+k) \cdots (\ell+1)} a_{n,\ell}, \quad \ell = 0, 1, \dots, n-1.$$

31 *Proof.* By Definition 1.3 we have

$$32 \quad (8) \quad \frac{d^k}{dz^k} (z - \xi)^k Q_{n;k}(z; \xi) = (n+k) \cdots (n+1) L_n(z).$$

35 Start with (8), use the power expansion (6) and taking into account the linearity of the derivative. If we
36 compare the power coefficients in these expressions the result holds. \square

37 By using this result we obtain the first main result.

38 **Theorem 3.2.** Let μ be a finite measure defined on the Borelian σ -algebra of \mathbb{C} such that it contains
39 an infinite number of points and let $(L_n(z))$ the system of monic orthogonal polynomials with respect
40 to μ . Let ξ be a fixed complex number. Let k be a positive integer.

41 All the zeros of $Q_{n;k}(z; \xi)$ are contained in the closed disk $\overline{D(0, |\xi| + (k+1)(1 + |\xi|))}$.

1 *Proof.* Since L_n is OPUC, then by [19] we know the zeros of $L_n(z)$ lie in \mathbb{D} . let us define $\omega := z - \xi$
 2 and, taking into account Lemma 3.1, let us consider the following polynomials:

$$3 \quad f_n(\omega) = L_n(z) = \sum_{\ell=0}^n \frac{a_{n,\ell}}{\binom{n}{\ell}} \binom{n}{\ell} \omega^\ell,$$

4
 5 and

$$6 \quad g_{n,k}(\omega) = \sum_{\ell=0}^n \frac{(n+k)\cdots(n+1)}{(\ell+k)\cdots(\ell+1)} \binom{n}{\ell} \omega^\ell = \sum_{\ell=0}^n \binom{n+k}{\ell+k} \omega^\ell = \frac{(n+k)\cdots(k+1)}{n!} F(-n, 1; k+1; -\omega),$$

7 where $F(a, b; c; z)$ is the Gauß function [4, 15.2.2].

8 The “composition” of f_n and g_n leads to

$$9 \quad h_{n,k}(\omega) = \sum_{\ell=0}^n \frac{b_{n,\ell;k}}{\binom{n}{\ell}} \binom{n}{\ell} \omega^\ell = Q_{n;k}(z; \xi).$$

10 Due to Theorem 3.2 in [5] we know that $g_{n;k}(z)$ has non-real zeros for n even and $n - 1$ non-real
 11 zeros for n odd. Moreover, it is known this function is defined for $|\omega| < 1$.

12 **Remark 3.3.** Observe that by using [1, Corollary 2] we obtain

$$13 \quad (9) \quad (1-z)^{n+k} F(n+k+1, k; k+1; z) = F(-n, 1; k+1; z) = \frac{n!}{(k+1)\cdots(k+n)} P_n^{(k, -k-n)}(1-2z),$$

14 where $P_n^{(\alpha, \beta)}(z)$ is the Jacobi polynomial of degree n and parameters α and β (see c.f. [4, 18.5.7] we
 15 can express the polynomial $g_{n;k}(z)$ in terms of the Jacobi polynomials.

16 **Remark 3.4.** Notice that if we take the k -th derivative of $\omega^k g_{n;k}(\omega)$ we obtain

$$17 \quad (10) \quad \frac{d^k}{d\omega^k} \omega^k g_{n;k}(\omega) = \sum_{\ell=0}^n (n+k)\cdots(n+1) \binom{n}{\ell} \omega^\ell = (n+k)\cdots(n+1)(1+\omega)^n.$$

18 Therefore for one side we know that if $z_{n,0}$ is a zero of $L_n(z)$ then $|z_{n,0}| \leq 1$, so $|w_{n,0}| \leq 1 + |\xi|$.
 19 On the other hand, since we know that if $w_{n,1}$ is a root of $g_{n;k}(z)$ then $|w_{n,1}| \leq R_{n,k}$. With these two
 20 inequalities we can claim that, using Szegő's Theorem, for any root of $h_{n;k}(z)$, namely $|z_{n,3}|$, and since
 21 $|\omega_{n,3}| = |z_{n,3} + \xi|$, the following inequality holds:

$$22 \quad (11) \quad |z_{n,3}| \leq (1 + |\xi|)R_{n,k} + |\xi|.$$

23 Since $P_1^{(k, -k-n)}(1+\omega) = \omega + k + 1$ and the zeros of the Jacobi polynomials $P_n^{(k, -k-n)}(1+\omega)$ tend to
 24 the circle $D(-1, 1)$ when $n \rightarrow \infty$ (see [10]), we can assume $R_{n,k} \leq k + 1$. Hence the result follows. \square

36 4. The examples

37 Along this section we will consider different examples which let us to explain why sometimes we
 38 can consider a ring shape region (strictly speaking) where the zeros are located in, and some other
 39 situations we can not. Of course the region depends on different parameters such as the value ξ , as
 40 well as the integers k and n , among others. The examples we are going to consider can be consider as
 41 canonical.

1 **4.1. First example. The Bernstein-Szegő Polynomials.** Let $\beta \in \mathbb{D}$, and let us consider the measure

2
3 (12)
$$d\mu(z) = \frac{1}{|z + \beta|^2} \frac{d\theta}{2\pi}.$$

4 Notice that if $\beta = r \exp(-i\phi)$, then (12) becomes [19, (1.6.2)]

5
6
$$d\mu(z) = P_r(\theta, \phi) \frac{d\theta}{2\pi},$$

7 where P_r is the Poisson kernel of

8
9
$$\int \Re(g(z)) z^{-n} \frac{d\theta}{2\pi}.$$

10 Let $(L_n(z))$ be the monic orthogonal polynomials with respect to μ . These polynomials can be
11 expressed as follows [7, 8]:

12
13 (13)
$$L_n(z) = z^n + \beta z^{n-1}, \quad n = 1, 2, \dots$$

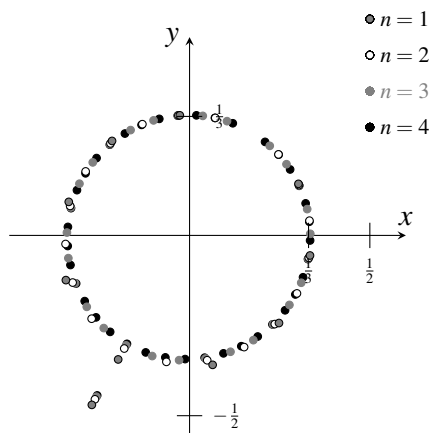
14 From this expression their first order monic polar polynomials are defined as:

15
16
17 (14)
$$Q_{n;1}(z; \xi) = (n+1) \frac{\int_{\xi}^z L_n(t) dt}{z - \xi} = \frac{z^n(nz + (n+1)\beta) - \xi^n(n\xi + (n+1)\beta)}{n(z - \xi)}.$$

18 Moreover, the second order monic polar polynomial of degree n is

19
20 (15)
$$Q_{n;2}(z; \xi) = \frac{z^{n+1}(nz + \beta n + 2\beta) + \xi^n(n(n+1)\xi^2 + n(n+2)\xi(\beta - z) - \beta(n+1)(n+2)z)}{n(n+1)(n+2)(z - \xi)^2}.$$

21 In the figures 1 and 2 we show the zeros of these polynomials under different settings (in the first case
22 $|\xi_1| + 2(1 + |\xi_1|) = 3$, and $|\xi_2| + 3(1 + |\xi_2|) = 13/3$ in the second one).
23
24



38
39
40
41
42
FIGURE 1. Zeros of $Q_{10n;1}(z; 1/3 \exp(i\pi/3))$ for $n = 1, 2, 3, 4$ in the window $[-0.5, 0.7] \times [-0.5, 0.7]$, with $\beta = 1/2 \exp(i\pi/3)$.

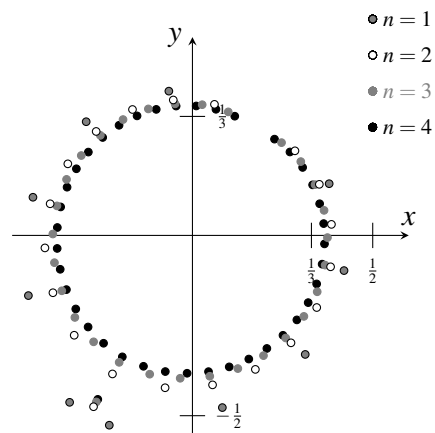


FIGURE 2. Zeros of $Q_{10n;2}(z; 1/3 \exp(i\pi/3))$ for $n = 1, 2, 3, 4$ in the window $[-0.5, 0.7] \times [-0.5, 0.7]$, with $\beta = 1/2 \exp(i\pi/3)$.

In order to show the Sendov's conjecture remains valid in the table 1 we present the maximum distance between each zero of $Q_{n,1}(z, \xi)$ and the closest zero of $Q'_{n,1}(z, \xi)$ for different values of n .

n	zero	distance	n	zero	distance
2	$-0.04549 - 0.59920i$	0.2602	2	$-0.89538 - 0.44530i$	0.5528
3	$0.25331 - 0.28868i$	0.41997	3	$0.40206 - 0.49827i$	0.85603
4	$0.32559 - 0.13863i$	0.65268	4	$0.47459 - 0.25270i$	0.89688
5	$0.34111 - 0.04380i$	0.68222	5	$-0.57262 - 0.15872i$	0.85890
6	$0.33895 + 0.01995i$	0.74277	6	$0.45849 - 0.01783i$	0.9069
7	$0.33098 + 0.06499i$	0.75514	7	$-0.35212 - 0.60990i$	0.88012
8	$0.32137 + 0.09813i$	0.77831	8	$0.41566 + 0.08718i$	0.90725
9	$-0.04905 + 0.33164i$	0.78565	9	$-0.32375 - 0.56075i$	0.88470
10	$-0.02743 + 0.33361i$	0.79638	10	$-0.06540 + 0.40007i$	0.90141
11	$-0.00958 + 0.33426i$	0.80117	11	$-0.04147 + 0.39636i$	0.88964
12	$0.00536 + 0.33412i$	0.80687	12	$-0.02178 + 0.39227i$	0.89539
13	$0.01804 + 0.33350i$	0.81013	13	$-0.00532 + 0.38809i$	0.88893
14	$0.02892 + 0.33261i$	0.81350	14	$0.00861 + 0.38400i$	0.89015
15	$0.03835 + 0.33155i$	0.81577	15	$0.02054 + 0.38006i$	0.88620
16	$0.04660 + 0.33042i$	0.81795	16	$0.03086 + 0.37633i$	0.88568
17	$0.05388 + 0.32925i$	0.81957	17	$0.03986 + 0.37280i$	0.88297
18	$0.06033 + 0.32808i$	0.82106	18	$0.04779 + 0.36949i$	0.88182
19	$0.06610 + 0.32693i$	0.82225	19	$0.05481 + 0.36639i$	0.87977
20	$0.07129 + 0.32581i$	0.82332	20	$0.06106 + 0.36348i$	0.87845

TABLE 1. For every $2 \leq n \leq 20$, $\beta = 1/2 \exp(i\pi/3)$, we obtain the zero of $Q_{n,1}(z; 1/3 \exp(i\pi/3))$ (left) and $Q_{n,2}(z; 1/3 \exp(i\pi/3))$ (right) which produces the maximum distance with respect to the zeros of their corresponding derivatives.

4.2. Second example. Fix $m \geq 0$. Let

$$d\mu_1(\theta) = \frac{d\theta}{2\pi} + m\delta(z-1),$$

where $z = \exp(i\theta)$ and

$$\int f(z) \delta(z-1) d\theta = f(1), \quad f \in \mathbb{P}[z].$$

The monic associated orthogonal polynomials with respect to μ_1 on \mathbb{T} are [2, 7]:

$$(16) \quad L_n(z) = z^n - \frac{m}{1+nm} \sum_{k=0}^{n-1} z^k = z^n - \frac{m}{1+nm} \frac{z^n - 1}{z - 1}, \quad n = 1, 2, \dots$$

From this expression their first order monic polar polynomials are defined as:

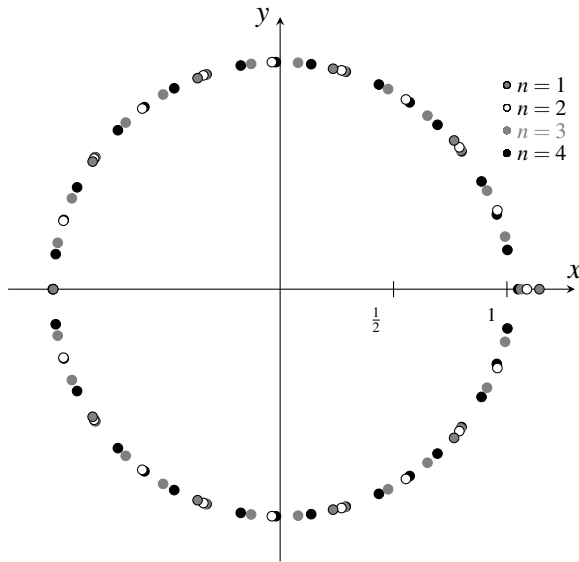
$$(17) \quad Q_{n,1}(z; \xi) = \frac{z^{n+1} - \xi^{n+1}}{z - \xi} - \frac{m(n+1)}{1+nm} \sum_{k=0}^{n-1} \frac{z^{k+1} - \xi^{k+1}}{(k+1)(z - \xi)}.$$

1 **Remark 4.1.** Notice that the zeros of these polynomials tend to accumulate around the unit circle \mathbb{T}
 2 whenever $|\xi| \leq 1$, and around the closed circle $\overline{D(0, |\xi|)}$ whenever $|\xi| > 1$. Moreover, since we added
 3 a mass point at $z = 1$ then it is expected that some of zeros the polar polynomials close to $z = 1$ move
 4 outside of such boundary.

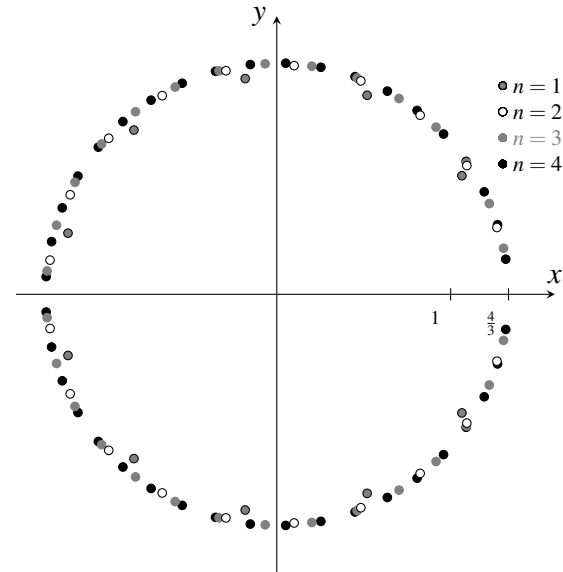
5 Moreover, due to Theorem 3.2 we know all the zeros of these polynomials lie inside of $\overline{D(0, 2 + 3|\xi|)}$.

6 In the figures 3 and 4 we show the zeros of these polynomials under different settings (in the first
 7 case $|\xi_1| + 2(1 + |\xi_1|) = 3$ and $|\xi_2| + 2(1 + |\xi_2|) = 6$ in the second one).
 8

9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25



26
27 **FIGURE 3.** Zeros of
 28 $Q_{10n;1}(z; 1/3)$ for $n = 1, 2, 3, 4$
 29 in the window $[-1, 1.2] \times$
 30 $[-1, 1]$, with $m = 2/3$.



31
32 **FIGURE 4.** Zeros
 33 of $Q_{10n;1}(z; 4/3)$ for
 34 $n = 1, 2, 3, 4$ in the win-
 35 dow $[-1.5, 1.5] \times [-1.5, 1.5]$,
 36 with $m = 2/3$.

33 In order to show the Sendov's conjecture remains valid in the table 2 we present the maximum
 34 distance between each zero of $Q_{n,1}(z, \xi)$ and the closet zero of $Q'_{n,1}(z, \xi)$ for different values of n .
 35

36 **4.3. Third Example.** Let

$$37 \quad d\mu_2(\theta) = |z - 1|^2 \frac{d\theta}{2\pi}.$$

38
 39 The monic associated orthogonal polynomials with respect to μ_2 on \mathbb{T} are [2, 7]:
 40

$$41 \quad (18) \quad L_n(z) = \sum_{k=0}^n \frac{k+1}{n+1} z^k = \frac{(n+1)z^{n+2} - (n+2)z^{n+1} + 1}{(n+1)(z-1)^2}, \quad n = 1, 2, \dots$$

42

	n	zero	distance	n	zero	distance
2	2	0.99163	0.9440	2	-0.45238 + 0.38021i	0.38021
3	3	1.1388	1.5309	3	-0.24822 + 0.77992i	1.2234
4	4	-0.97769	1.5490	4	0.13378 + 0.95290i	1.6349
5	5	1.1848	1.7297	5	0.44282 + 0.96439i	1.7519
6	6	-0.99320	1.7641	6	-0.28585 - 1.04862i	1.9307
7	7	1.1711	1.7968	7	-0.81931 - 0.79845i	2.0409
8	8	-0.99834	1.8563	8	-0.60642 - 1.03052i	2.1514
9	9	-0.94557 - 0.32389i	1.8579	9	-0.99493 - 0.71892i	2.2245
10	10	-1.0005	1.9036	10	-0.82147 - 0.94486i	2.2841
11	11	-0.96404 - 0.26952i	1.9012	11	-1.09907 - 0.63389i	2.3315
12	12	-1.0015	1.9310	12	-0.95942 - 0.84975i	2.3671
13	13	-0.97483 - 0.23056i	1.9275	13	-1.16310 - 0.56021i	2.3993
14	14	-1.0020	1.9483	14	-1.05017 - 0.76336i	2.4222
15	15	-0.98163 - 0.20134i	1.9446	15	-1.20452 - 0.49917i	2.4452
16	16	-1.0022	1.9598	16	-1.11214 - 0.68904i	2.4609
17	17	-0.98617 - 0.17864i	1.9563	17	-1.23260 - 0.44888i	2.4779
18	18	-1.0023	1.9679	18	-1.15604 - 0.62599i	2.4892
19	19	-0.98934 - 0.16051i	1.9647	19	-1.25243 - 0.40715i	2.5022
20	20	-1.0023	1.9738	20	-1.18814 - 0.57248i	2.5107

TABLE 2. For every $2 \leq n \leq 20$, $m = 2/3$, we obtain the zero of $Q_{n,1}(z; 1/3)$ (left) and $Q_{n,1}(z; 4/3)$ (right) which produces the maximum distance with respect to the zeros of their corresponding derivatives.

From this expression their first order monic polar polynomials are defined as:

$$(19) \quad Q_{n,1}(z; \xi) = \frac{z(z^{n+1} - 1)(\xi - 1) - \xi(\xi^{n+1} - 1)(z - 1)}{(\xi - 1)(z - \xi)(z - 1)}, \quad z \neq 1, z \neq \xi.$$

In the figures 5 and 6 we show the zeros for the $k = 1$ and $k = 4$ polar polynomials. In the figures 5 and 6 we show the zeros of these polynomials under different settings (in the first case $|\xi_1| + 2(1 + |\xi|_1) = 3$ and $|\xi_2| + 5(1 + |\xi_2|) = 13$ in the second one).

In the $k = 4$ case, and due of the length and difficulty of these expressions for the polynomials, are not presented in the manuscript.

References

[1] K. Boggs and P. Duren. Zeros of hypergeometric functions. *Comput. Methods Funct. Theory*, 1(1, [On table of contents: 2002]):275–287, 2001.

[2] P. Borwein and T. Erdélyi. *Polynomials and polynomial inequalities*, volume 161 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[3] B. Datt and N. K. Govil. On the location of the zeros of a polynomial. *J. Approx. Theory*, 24:78–82, 1978.

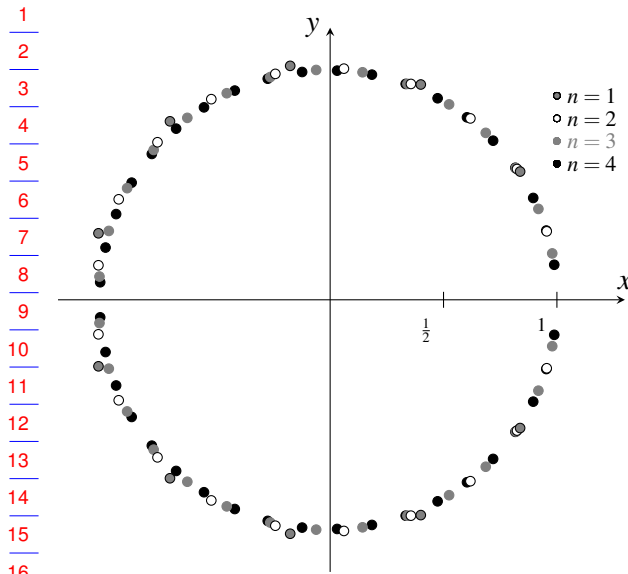


FIGURE 5. Zeros of $Q_{10n;1}(z; 1/3)$ for $n = 1, 2, 3, 4$ in the window $[-1, 1.2] \times [-1, 1]$.

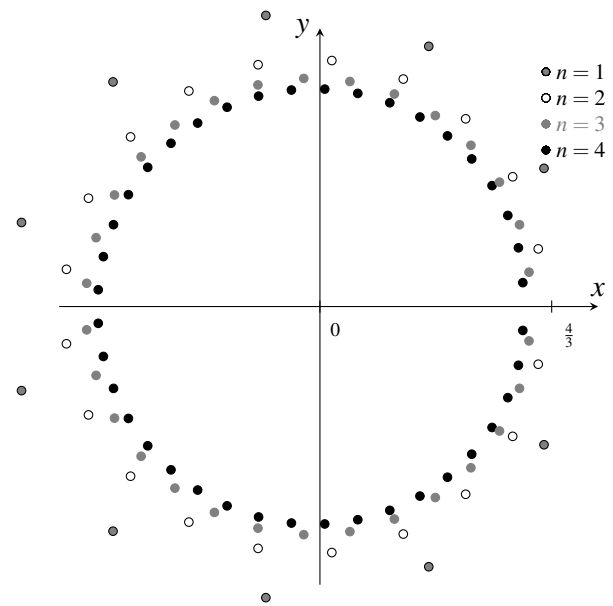


FIGURE 6. Zeros of $Q_{10n;4}(z; 4/3)$ for $n = 1, 2, 3, 4$ in the window $[-1.5, 1.5] \times [-1.5, 1.5]$.

- [4] *NIST Digital Library of Mathematical Functions*. <https://dlmf.nist.gov/>, Release 1.1.9 of 2023-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [5] K. Driver and K. Jordaan. Zeros of the hypergeometric polynomial $F(-n, b; c; z)$. *Proceedings of Algorithms for Approximations IV*, pages 436–445, 2002.
- [6] A. Fundora Rolo, H. Pijeira Cabrera, and W. Urbina Romero. Asymptotic behavior of orthogonal polynomials primitives. In *Margarita mathematica*, pages 627–633. Univ. La Rioja, Logroño, 2001.
- [7] L. Garza and F. Marcellán. Verblunsky parameters and linear spectral transformations. *Methods Appl. Anal.*, 16(1):69–85, 2009.
- [8] E. Godoy and F. Marcellán. Orthogonal polynomials and rational modifications of measures. *Canad. J. Math.*, 45(5):930–943, 1993.
- [9] J. H. Grace. On the zeros of a polynomial. *Proc. Camb. Philos. Soc.*, 11:352–356, 1902.
- [10] A. B. J. Kuijlaars, A. Martínez-Finkelshtein, and R. Orive. Orthogonality of Jacobi polynomials with general parameters. *Electron. Trans. Numer. Anal.*, 19:1–17, 2005.
- [11] E. Levin and D. S. Lubinsky. Asymptotics of derivatives of orthogonal polynomials on the real line. *Acta Math. Hungar.*, 118(1-2):115–127, 2008.
- [12] F. Lucas. Theorems on algebraic equations. *C. R. Acad. Sci., Paris*, 78:431–433, 1874.
- [13] M. Marden. *Geometry of polynomials*. Mathematical Surveys, No. 3. American Mathematical Society, Providence, R.I., second edition, 1966.
- [14] M. Marden. Conjectures on the critical points of a polynomial. *Amer. Math. Monthly*, 90(4):267–276, 1983.
- [15] M. J. Miller. On Sendov’s conjecture for roots near the unit circle. *J. Math. Anal. Appl.*, 175(2):632–639, 1993.
- [16] G. V. Milovanović and T. M. Rassias. Inequalities for polynomial zeros. In *Survey on classical inequalities*, volume 517 of *Math. Appl.*, pages 165–202. Kluwer Acad. Publ., Dordrecht, 2000.

- 1 [17] P. Nevai and V. Totik. Orthogonal polynomials and their zeros. *Acta Sci. Math. (Szeged)*, 53(1-2):99–104, 1989.
- 2 [18] H. Pijeira Cabrera, J. Y. Bello Cruz, and W. Urbina Romero. On polar Legendre polynomials. *Rocky Mountain J. Math.*,
3 40(6):2025–2036, 2010.
- 4 [19] B. Simon. *Orthogonal polynomials on the unit circle. Part I*, volume 54 of *American Mathematical Society Colloquium
5 Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory.
- 6 [20] G. Szegő. Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen. *Math. Z.*,
7 13(1):28–55, 1922.
- 8 [21] G. Szegő. *Orthogonal Polynomials*. Amer. Math. Soc. Coll. Pub. 23 AMS, Providence, Rhode Island, 4 edition, 1975.

8 DPTO. MÉTODOS CUANTITATIVOS, UNIVERSIDAD LOYOLA ANDALUCÍA, DOS HERMANAS, SEVILLE, SPAIN
9 *Email address:* rscosa@gmail.com

10
11 DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT SCIENCES, UNIVERSITY OF HAMA LAKHDAR OF ELOUED,
12 ALGERIA.
13 *Email address:* rehouma-abdelhamid@univ-eloued.dz

14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42