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Integral representations of a generalized linear Hermite functional

Roberto S. Costas–Santos

Department of Quantitative Methods, Universidad Loyola Andalucía, Sevilla, Spain; rscosa@gmail.com

Version July 18, 2023 submitted to Preprints

Abstract: In this paper, we find new integral representations for the *generalized Hermite linear functional* in the real line and the complex plane. As an application, new integral representations for the Euler Gamma function are given.

Keywords: Integral representation; Hermite functions; Generalized Hermite linear functional; Gamma function

MSC: 33C45, 42C05 (Primary); 30E20, 33B15 (Secondary)

1. Introduction

The integral representation of special functions provides an alternative way to express these functions in terms of integrals involving other functions. Often they involve a weight function and a kernel function related to the specific special function being considered. The weight function appears as a factor in the integral and reflects the orthogonality property of the associated orthogonal polynomials, and the kernel function represents the additional dependence.

The integral representation allows us to express special functions as infinite series or integrals involving some classical orthogonal polynomials. This connection arises from the fact that the orthogonality condition is satisfied by classical orthogonal polynomials naturally leads to the appearance of these polynomials in the integral representation of special functions. In this work, we are going to consider the Hermite polynomials.

The hypergeometric functions, which have applications in many areas, including mathematical physics and combinatorics can be represented in terms of integrals involving other hypergeometric functions and classical orthogonal polynomials like the Jacobi, Hermite, and Laguerre polynomials, which can be expressed as hypergeometric series (see c.f. [1] and [7, §16.5]).

For a detailed history of the subject of integral representations for hypergeometric series and basic hypergeometric functions (which is a natural extension of the hypergeometric series), see [3] and [4, Chapter 4].

R. Sfaxi has established in [9], by means of a linear isomorphism, the so-called *intertwining operator* on polynomials, a relationship between the ordinary Hermite polynomials and their analog nonsingular and of Laguerre-Hahn with class zero. Among others, the author has put in value an important linear functional, namely *the generalized Hermite linear functional*, denoted by $\mathscr{G}_H(\tau)$ of index $\tau \in \mathbb{C}$, with $\tau \neq -n,\ n \geq 1$, where their moments are given by

$$\big(\mathscr{G}_H(\tau)\big)_n := \langle \mathscr{G}_H(\tau), x^n \rangle = \begin{cases} \dfrac{(\tau+1)_{2k}}{k!2^{2k}}, \text{ if } n = 2k, \\ \qquad 0, \text{ if } n = 2k+1, \end{cases} \tag{1.1}$$

where $(a)_n$ is the Pochhammer symbol, defined as

$$(a)_0 := 1, \quad (a)_k := a(a+1)\cdots(a+k-1), \quad a \in \mathbb{C} \setminus \{0\},\ k = 1,2,3,\dots,$$

thus $\mathscr{G}_H(\tau)$ is symmetric and monic, i.e., $\big(\mathscr{G}_H(\tau)\big)_0 = 1$.

Observe that setting $\tau = 0$ in (1.1) we recover the Hermite linear functional, i.e., $\mathscr{G}_H \equiv \mathscr{G}_H(0)$, that is well-known by its integral representation

$$\langle \mathscr{G}_H, p \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} p(x) e^{-x^2} dx, \quad p \in \mathbb{P}. \tag{1.2}$$

So we can write

$$\big(\mathscr{G}_H(\tau)\big)_n = \frac{(\tau+1)_n}{(1)_n} (\mathscr{G}_H)_n, \quad n = 0, 1, \ldots$$

Note that the linear functional $\mathscr{G}_H$ is classical since it is quasi-definite and satisfies the Pearson equation

$$\mathscr{G}_H' + 2x\mathscr{G}_H = 0. \tag{1.3}$$

Taking this into account the following result holds.

Lemma 1.1. *For any $\tau \in \mathbb{C}$, the linear functional $\mathscr{G}_H(\tau)$ fulfills the difference equation*

$$\big(x^2 \mathscr{G}_H(\tau)\big)'' + \big(2x(x^2 - \tau - 2)\mathscr{G}_H(\tau)\big)' + \big(-4x^2 + (\tau+1)(\tau+2)\big)\mathscr{G}_H(\tau) = 0.$$

Proof. Let $\tau \in \mathbb{C}$, if we define the linear functional $\mathscr{E}(\tau)$ as

$$\mathscr{E}(\tau) := \big(x^2 \mathscr{G}_H(\tau)\big)'' + \big(2x(x^2 - \tau - 2)\mathscr{G}_H(\tau)\big)' + \big(-4x^2 + (\tau+1)(\tau+2)\big)\mathscr{G}_H(\tau).$$

Then, for $n \geq 0$, one gets

$$\big(\mathscr{E}(\tau)\big)_n = -2(n+2)\big(\mathscr{G}_H(\tau)\big)_{n+2} + (n+\tau+2)(n+\tau+1)\big(\mathscr{G}_H(\tau)\big)_n. \tag{1.4}$$

Since $\mathscr{G}_H(\tau)$ is symmetric, then $\big(\mathscr{E}(\tau)\big)_{2k+1} = 0$, for every $k \geq 0$. On the other hand, setting $n = 2k$ in (1.4) and taking into account (1.1), we get for $k \geq 0$,

$$\begin{aligned}
\big(\mathscr{E}(\tau)\big)_{2k} &= -4(k+1)\big(\mathscr{G}_H(\tau)\big)_{2k+2} + (2k+\tau+2)(2k+\tau+1)\big(\mathscr{G}_H(\tau)\big)_{2k} \\
&= -\frac{(\tau+1)_{2k+2}}{k!2^{2k}} + \frac{(2k+1+\tau+1)(2k+\tau+1)(\tau+1)_{2k}}{k!2^{2k}} \\
&= 0.
\end{aligned}$$

Therefore $\big(\mathscr{E}(\tau)\big)_n = 0$ for all $n = 0, 1, \ldots$. Hence the result holds. □

Our purpose in this work is to provide integral representations for the linear functional $\mathscr{G}_H(\tau)$, either on the real axis, or on the complex plane.
More precisely, the problem consists in to determinate a weight function $G_H(\bullet; \tau)$ such that

$$\langle \mathscr{G}_H(\tau), p \rangle = \int_\Omega p(x) G_H(x; \tau) dx, \quad p \in \mathbb{P},$$

where Ω is an interval in the real line, or a contour in the complex plane.

The paper is organized as follows. In the next section, some preliminaries and notation. In Sections 3 and 4, integral representations in the real line and in the complex plane, respectively, are given. As an application of the previous results, in Section 5, some new integral representations for the Euler Gamma function are given.

2. preliminaries and notation

Let $\mathbb{P}$ be the vector space of polynomials with complex coefficients and let $\mathbb{P}'$ be its dual space. We denote by $\langle u, f \rangle$ the action of the linear functional $u \in \mathbb{P}'$ on the polynomial $f \in \mathbb{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of u.

Definition 2.1. *A linear functional u is called* symmetric *if $(u)_{2n+1} = 0$, for all $n = 0, 1, \ldots$, and it is called* monic *if $(u)_0 = 1$.*

In fact, for any $\tau \in \mathbb{C}$, the linear functional $\mathscr{G}_H(\tau)$ is symmetric (see (1.1)) which allow us to suppose the weight function $G_H(\bullet;\tau)$ is even, i.e., it can be written as $G_H(x;\tau) = U(|x|;\tau)$, where $U(\bullet;\tau)$ is a function defined on $(0,\infty)$. In fact, this is a direct consequence of the following result.

Lemma 2.2. *Let $\mathscr{L}$ be a symmetric linear function having an integral representation. Then, there exists a function U defined on $(0,\infty)$ such that*

$$\langle \mathscr{L}, p \rangle = \int_{-\infty}^{\infty} p(x)U(|x|)dx.$$

Proof. From the assumption there exists a function L, defined on $(-\infty,\infty)$, such that

$$\langle \mathscr{L}, p \rangle = \int_{-\infty}^{\infty} p(x)L(x)dx.$$

Let us introduce the following two functions, defined on $(0,\infty)$, as follows:

$$U(x) = \frac{L(x) + L(-x)}{2}, \quad V(x) = \begin{cases} \dfrac{L(x) - L(-x)}{2x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

A straightforward calculation gives that $L(x) = U(|x|) + xV(|x|)$, for all $x \in \mathbb{R}$. Moreover, since $x^{2n+1}V(|x|)$ is an odd function we have

$$(\mathscr{L})_{2n} = \int_{-\infty}^{\infty} x^{2n}U(|x|)dx + \int_{-\infty}^{\infty} x^{2n+1}V(|x|)dx = \int_{-\infty}^{\infty} x^{2n}U(|x|)dx.$$

On the other hand, since $\mathscr{L}$ is symmetric and $x^{2n+1}U(|x|)$ is an odd function, we get

$$(\mathscr{L})_{2n+1} = \int_{-\infty}^{\infty} x^{2n+1}U(|x|)dx = 0.$$

Therefore, for any polynomial $p \in \mathbb{P}$,

$$\langle \mathscr{L}, p \rangle = \int_{-\infty}^{\infty} p(x)U(|x|)dx.$$

$\square$

Next result related with hypergeometric functions will be useful later.

Lemma 2.3. *[5,6] The following formulae hold:*

1. *If $\Re(\alpha) > 0$ and $\Re(s) > 0$, then*

$$\int_0^{\infty} t^{\alpha-1}{}_1F_1(a_1;b_1;t)e^{-st}dt = \frac{\Gamma(\alpha)}{s^{\alpha}}\,{}_2F_1(a_1,\alpha;b_1;1/s). \tag{2.1}$$

2. *If $\Re(c-a-b) > 0$, then*

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.2}$$

where

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \qquad {}_1F_1(a;b;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!}.$$

In the sequel we will denote by $H_\tau(x)$ the *Hermite function (of degree τ)*, which can be represented in terms of the confluent hypergeometric function ${}_1F_1$ as follows [6]:

$$H_\tau(x) = 2^\tau \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\tau}{2})} {}_1F_1\left(-\frac{\tau}{2}; \frac{1}{2}; x^2\right) + 2^\tau x \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\tau}{2})} {}_1F_1\left(\frac{1-\tau}{2}; \frac{3}{2}; x^2\right). \tag{2.3}$$

3. Integral representation on $\mathbb{R}$

In the following result, we present a new definite integration formulae involving the Hermite functions.

Lemma 3.1. *For any $(z,\tau) \in \mathbb{C}^2$, with $\Re(z) > -1$, the following formulae hold:*

$$\int_0^\infty x^z H_\tau(x) e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{z-\tau+1}} \frac{\Gamma(z+1)}{\Gamma(\frac{z-\tau}{2}+1)}, \tag{3.1}$$

$$\int_{-\infty}^\infty |x|^z H_\tau(|x|) e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{z-\tau}} \frac{\Gamma(z+1)}{\Gamma(\frac{z-\tau}{2}+1)}. \tag{3.2}$$

Proof. Since the function $|x|^\nu H_\tau(|x|)e^{-x^2}$ is even, then it is enough to prove (3.1).

Let us fix $\tau \in \mathbb{C}$, with $\Re(\tau) > -1$. For any $z \in \mathbb{C}$, with $-1 < \Re(z) < \Re(\tau)$, let us consider the following integral:

$$\Lambda(z) := \int_0^\infty x^z H_\tau(x) e^{-x^2} dx.$$

Using (2.3), the previous integral can be written as

$$\Lambda(z) = 2^\tau \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\tau}{2})} \Pi(z) + 2^\tau \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\tau}{2})} \Omega(z), \tag{3.3}$$

where

$$\begin{aligned} \Pi(z) &:= \int_0^\infty x^z {}_1F_1\left(-\frac{\tau}{2}; \frac{1}{2}; x^2\right) e^{-x^2} dx, \\ \Omega(z) &:= \int_0^\infty x^{z+1} {}_1F_1\left(\frac{1-\tau}{2}; \frac{3}{2}; x^2\right) e^{-x^2} dx. \end{aligned}$$

By changing the variable of integration, by setting $t = x^2$, and using (2.1), with $s = 1$, $\alpha = (z+1)/2$, $a_1 = -\tau/2$, and $b_1 = 1/2$, we obtain

$$\Pi(z) = \frac{1}{2} \Gamma\left(\frac{z+1}{2}\right) {}_2F_1\left(-\frac{\tau}{2}, \frac{z+1}{2}; \frac{1}{2}; 1\right).$$

Again, with (2.1), where $s = 1$, $\alpha = (z+2)/2$, $a_1 = (1-\tau)/2$, and $b_1 = 3/2$, we get

$$\Omega(z) = \frac{1}{2} \Gamma\left(\frac{z+2}{2}\right) {}_2F_1\left(\frac{1-\tau}{2}, \frac{z+2}{2}; \frac{3}{2}; 1\right).$$

Since $\Re(z) < \Re(\tau)$, by using (2.2) $\Pi(z)$ and $\Omega(z)$ can be written as

$$\begin{aligned} \Pi(z) &= \frac{\Gamma(\frac{z+1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{\tau-z}{2})}{2\Gamma(\frac{1+\tau}{2})\Gamma(-\frac{z}{2})}, \\ \Omega(z) &= \frac{\Gamma(\frac{z+2}{2})\Gamma(\frac{3}{2})\Gamma(\frac{\tau-z}{2})}{2\Gamma(\frac{2+\tau}{2})\Gamma(\frac{1-z}{2})}. \end{aligned}$$

Therefore, taking into account $\Gamma(\frac{1}{2})^2 = -\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2}) = \pi$, the expression (3.3) can be rewritten as follows:

$$\Lambda(z) = \frac{2^{\tau-1}\pi\Gamma(\frac{\tau-z}{2})}{\Gamma(-\frac{z}{2})\Gamma(\frac{1-z}{2})}\Big(U(z,\tau) - U(z+1,\tau+1)\Big),$$

where

$$U(z,\tau) = \frac{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}{\Gamma(\frac{1+\tau}{2})\Gamma(\frac{1-\tau}{2})}.$$

Using the duplication formula

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin(\pi u)},$$

a straightforward calculation leads to

$$U(z,\tau) = \frac{\cos(\frac{\pi}{2}\tau)}{\cos(\frac{\pi}{2}z)}, \quad U(z+1,\tau+1) = \frac{\sin(\frac{\pi}{2}\tau)}{\sin(\frac{\pi}{2}z)}.$$

Then,

$$\Lambda(z) = -\frac{2^{\tau}\pi\Gamma(\frac{\tau-z}{2})}{\Gamma(-\frac{z}{2})\Gamma(\frac{1-z}{2})}\frac{\sin\left(\frac{\pi}{2}(\tau-z)\right)}{\sin(\pi z)},$$

so, by using the Gauss-Legendre multiplication formula,

$$\Gamma(u)\Gamma(u+\tfrac{1}{2}) = 2^{1-2u}\sqrt{\pi}\,\Gamma(2u),$$

and, again, with the duplication formula, we get

$$\Lambda(z) = \frac{\sqrt{\pi}}{2^{z-\tau+1}}\frac{\Gamma(z+1)}{\Gamma(1+\frac{z-\tau}{2})}.$$

For this proof we assumed the conditions $-1 < \Re(z) < \Re(\tau)$, then the integral $\Lambda(z)$ converges exponentially to zero when $\tau \to \infty$. Hence, by analytic continuation, (3.3) is valid for each $(\tau, z) \in \mathbb{C}^2$, with $\Re(z) > -1$. $\square$

Remark 3.2. *Note that the above result also covers the $z = \tau$ case. In fact, if τ =0, 1, ... this identity represents the property of orthogonality for the monic Hermite polynomials.*

As a consequence we have the following result:

Corollary 3.3. *For any $\tau \in \mathbb{C}$, with $\Re(\tau) > -1$, then the following formulae hold:*

$$\int_0^{\infty} x^{2n+\tau} H_\tau(x) e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{2n+1}}\frac{\Gamma(2n+\tau+1)}{\Gamma(n+1)}, \tag{3.4}$$

$$\int_{-\infty}^{\infty} x^{2n}|x|^{\tau} H_\tau(|x|) e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{2n}}\frac{\Gamma(2n+\tau+1)}{\Gamma(n+1)}. \tag{3.5}$$

Theorem 3.4. *For any $\tau \in \mathbb{C}$, with $\Re(\tau) > -1$, then the linear functional $\mathscr{G}_H(\tau)$ has the following integral representation:*

$$\langle \mathscr{G}_H(\tau), p\rangle = \frac{1}{\sqrt{\pi}\,\Gamma(\tau+1)}\int_{-\infty}^{\infty} p(x)|x|^{\tau} H_\tau(|x|) e^{-x^2} dx, \quad p \in \mathbb{P}, \tag{3.6}$$

where H_τ is the Hermite function (of degree τ).

Proof. Due the equation (1.1) and corollary 3.3, then

$$\begin{aligned}
\big(\mathscr{G}_H(\tau)\big)_{2n} &= \frac{(\tau+1)_{2n}}{n!2^{2n}} = \frac{\Gamma(2n+\tau+1)}{2^{2n}\Gamma(n+1)\Gamma(\tau+1)} \\
&= \frac{1}{\sqrt{\pi}\,\Gamma(\tau+1)} \int_{-\infty}^{\infty} x^{2n}|x|^{\tau} H_\tau(|x|)e^{-x^2}dx, \\
\big(\mathscr{G}_H(\tau)\big)_{2n+1} &= 0 = \frac{1}{\sqrt{\pi}\,\Gamma(\tau+1)} \int_{-\infty}^{\infty} x^{2n+1}|x|^{\tau} H_\tau(|x|)e^{-x^2}dx.
\end{aligned}$$

Therefore, one has

$$\big(\mathscr{G}_H(\tau)\big)_{n} = \frac{1}{\sqrt{\pi}\,\Gamma(\tau+1)} \int_{-\infty}^{\infty} x^{n}|x|^{\tau} H_\tau(|x|)e^{-x^2}dx, \qquad n = 0,1,\ldots$$

Consequently, for any polynomial $p \in \mathbb{P}$,

$$\langle \mathscr{G}_H(\tau), p\rangle = \frac{1}{\sqrt{\pi}\,\Gamma(\tau+1)} \int_{-\infty}^{\infty} p(x)|x|^{\tau} H_\tau(|x|)e^{-x^2}dx.$$

□

Observe that if we set $n = 0$ in (3.4) we get a new integral representation for the Euler Gamma function. In fact, for any $\tau \in \mathbb{C}$, with $\Re(\tau) > -1$,

$$\Gamma(\tau+1) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} x^{\tau} H_\tau(x)e^{-x^2}dx, \tag{3.7}$$

$$\Gamma(\tau+1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |x|^{\tau} H_\tau(|x|)e^{-x^2}dx. \tag{3.8}$$

4. Integral representation on the complex plane

Theorem 4.1. *For any $\tau \in \mathbb{C}$, the following identities hold:*

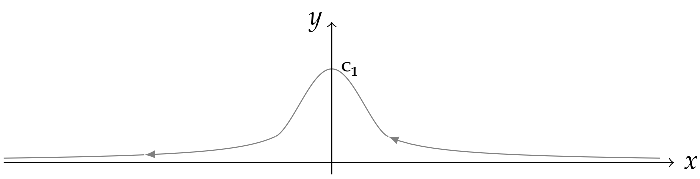
i)

$$\int_{\mathbf{C_1}} \zeta^{2n+1}|\zeta|^{\tau} H_\tau(|\zeta|)e^{-\zeta^2}d\zeta = 0, \quad n = 0,1,\ldots$$

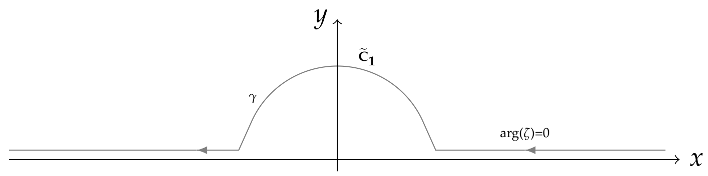
ii) For any $n \in \mathbb{N}$, so that $\tau + 2n + 1$ is not a negative integer, we have

$$\int_{\mathbf{C_1}} \zeta^{2n}|\zeta|^{\tau} H_\tau(|\zeta|)e^{-\zeta^2}d\zeta = -\frac{\sqrt{\pi}}{2^{2n}}\frac{\Gamma(2n+\tau+1)}{\Gamma(n+1)}, \quad n = 0,1,\ldots$$

where $\mathbf{C_1}$ is the following contour in the complex plane:



Proof. We deform $\mathbf{C_1}$ into a contour $\tilde{\mathrm{C}}_1$ consisting of two straight lines and a circle as follows:



where $\gamma := \{\zeta \in \mathbb{C} : \Im(\zeta) > 0, |\zeta| = \epsilon\}$, being $\epsilon > 0$.

Now, for each integer $n \geq 0$ and $\tau \in \mathbb{C}$, we define

$$\begin{aligned} I_n(\tau) := & \int_{\tilde{\mathbf{C}}_1} \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta = \int_\infty^\epsilon \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta \\ & + \int_\gamma \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta + \int_{-\epsilon}^{-\infty} \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta. \end{aligned}$$

So, if $\Re(\tau) > -n-1$, after a direct computation, we get

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\infty^\epsilon \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta &= -\int_0^\infty x^{n+\tau} H_\tau(x) e^{-x^2} dx, \\ \lim_{\epsilon \to 0} \int_{-\epsilon}^{-\infty} \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta &= -(-1)^n \int_0^\infty x^{n+\tau} H_\tau(x) e^{-x^2} dx. \end{aligned}$$

For the middle integral, we obtain

$$\begin{aligned} \Big| \int_\gamma \zeta^n |\zeta|^\tau H_\tau(|z|) e^{-\zeta^2} d\zeta \Big| &= \Big| \int_0^\pi \epsilon^n e^{in\theta} \epsilon^\tau H_\tau(\epsilon) e^{-\epsilon^2 e^{2i\theta}} \epsilon i e^{i\theta} d\theta \Big| \\ &\leq \epsilon^{n+\Re(\tau)+1} \int_0^\pi |H_\tau(\epsilon)| e^{-\epsilon^2 \cos(2\theta)} d\theta, \end{aligned}$$

knowing that $H_\tau(0) = 2^\tau \sqrt{\pi}/\Gamma(\frac{1-\tau}{2})$, it is straightforward to see that

$$\lim_{\epsilon \to 0} \int_\gamma \zeta^n |\zeta|^\tau H_\tau(|\zeta|) e^{-\zeta^2} d\zeta = 0.$$

Therefore, for each $n \geq 0$ and $\tau \in \mathbb{C}$, such that $\Re(\tau) > -n-1$, we have

$$I_n(\tau) = -\big((-1)^n + 1\big) \int_0^\infty x^{n+\tau} H_\tau(x) e^{-x^2} dx.$$

Then $I_{2n+1}(\tau) = 0$ for all $n \geq 0$. Notice that for the proof of i) we assumed $\Re(\tau) > -n-1$, but the integral converges exponentially when $\tau \to \infty$, and therefore it exists for all τ. Hence i) holds by analytic continuation for any $\tau \in \mathbb{C}$.
On the other hand, by using (3.4), it follows that

$$I_{2n}(\tau) = -\frac{\sqrt{\pi}}{2^{2n}} \frac{\Gamma(2n+\tau+1)}{\Gamma(n+1)}.$$

Hence ii) holds, for the same reason already quoted and by analytic continuation of $\tau \in \mathbb{C}$, except when $2n + \tau + 1$ is a negative integer, where the function Γ is undefined. □

As a consequence, we have the following result.

Theorem 4.2. *For any $\tau \in \mathbb{C}$, with $-\tau \notin \mathbb{N}$, then the linear functional $\mathscr{G}_H(\tau)$ has the following integral representation:*

$$\langle \mathscr{G}_H(\tau), p \rangle = -\frac{1}{\sqrt{\pi}\Gamma(\tau+1)} \int_{\mathbf{C}_1} p(x) |x|^\tau H_\tau(|x|) e^{-x^2} dx, \quad p \in \mathbb{P}, \tag{4.1}$$

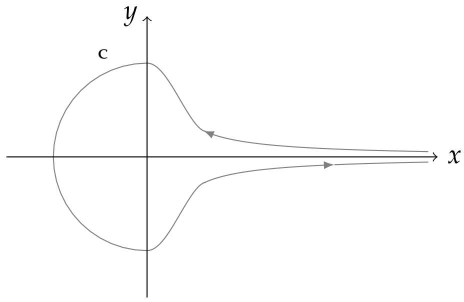
where H_τ is the Hermite function (of degree τ).

Using an analog idea allows us to formulate another integral representation for the gamma function in the complex plane, by using a different contour.

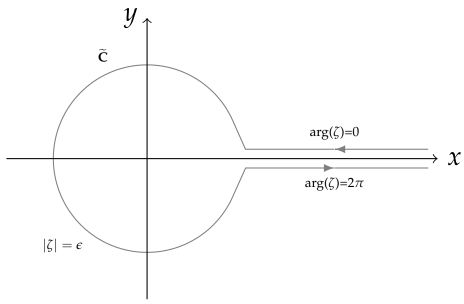
Theorem 4.3. *For any $\tau \in \mathbb{C}$, with $-\tau \notin \mathbb{N}$, then the Euler's Gamma function satisfies the following integral representation:*

$$\Gamma(\tau+1) = \frac{2}{\sqrt{\pi}(e^{2\pi i\tau}-1)} \int_{\mathbf{C}} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta, \tag{4.2}$$

where **C** *is the following contour in the complex plane:*



Proof. We deform **C** into a contour $\tilde{\mathbf{C}}$ consisting of two straight lines and a circle as follows:



We let

$$J(\tau) = \int_{\tilde{\mathbf{C}}} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta$$

Then

$$J(\tau) = \int_{\infty}^{\epsilon} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta + \int_{|\zeta|=\epsilon} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta + \int_{\epsilon}^{\infty} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta,$$

and if $\Re(\tau) > -1$ in a direct way we obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{\infty}^{\epsilon} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta &= -\frac{\sqrt{\pi}}{2}\Gamma(\tau+1), \\ \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta &= e^{2\pi i\tau}\frac{\sqrt{\pi}}{2}\Gamma(\tau+1). \end{aligned}$$

For the middle integral we obtain

$$\begin{aligned} \Big| \int_{|\zeta|=\epsilon} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta \Big| &= \left| \int_0^{2\pi} (\epsilon e^{i\theta})^{\tau} H_{\tau}(\epsilon e^{i\theta}) e^{-\epsilon^2 e^{2i\theta}} \epsilon i e^{i\theta} d\theta \right| \\ &\leq \epsilon^{\Re(\tau)+1} \int_0^{2\pi} |H_{\tau}(\epsilon e^{i\theta})| e^{-\epsilon^2\cos(2\theta)-\theta\left(\Im(\tau)+1\right)} d\theta, \end{aligned}$$

thus

$$\lim_{\epsilon\to 0} \int_{|\zeta|=\epsilon} \zeta^{\tau} H_{\tau}(\zeta) e^{-\zeta^2} d\zeta = 0.$$

Finally

$$J(\tau) = (e^{2\pi i\tau} - 1)\frac{\sqrt{\pi}}{2}\Gamma(\tau+1),$$

hence the result holds. In the proof we have assumed that $\Re(\tau) > -1$, but the integral (4.2) converges exponentially at infinity, and therefore it exists for all τ. In fact, by analytic continuation the result is valid for every complex τ, except for the negative integers, where the denominator vanishes. □

In addition, from the last representation, we obtain the following one:

$$\Gamma(\tau+1) = \frac{1}{i\sqrt{\pi}\sin(\pi\tau)}\int_{\mathbf{C}}(-\zeta)^{\tau}H_{\tau}(\zeta)e^{-\zeta^2}d\zeta.$$

In the last result we show a representation for the reciprocal of $\Gamma(\tau+1)$.

Theorem 4.4.

$$\frac{1}{\Gamma(\tau+1)} = -i\pi^{-\frac{3}{2}}\int_{\mathbf{C}}(-\zeta)^{-1-\tau}H_{-1-\tau}(\zeta)e^{-\zeta^2}d\zeta.$$

*This representation is valid for all τ and **C** is the same contour as in the previous theorem.*

Proof. By the last representation, one has

$$\begin{aligned}\Gamma(-\tau) &= \frac{1}{i\sqrt{\pi}\sin(\pi\tau)}\int_{\mathbf{C}}(-\zeta)^{-1-\tau}H_{-1-\tau}(\zeta)e^{-\zeta^2}d\zeta \\ &= \frac{\Gamma(\tau+1)\Gamma(-\tau)}{i\pi^{\frac{3}{2}}}\int_{\mathbf{C}}(-\zeta)^{-1-\tau}H_{-1-\tau}(\zeta)e^{-\zeta^2}d\zeta.\end{aligned}$$

This leads to the desired result. □

5. Conclusions

We have obtained integral representations of a generalized linear Hermite functional, which is among the natural extensions of the linear Hermite functional, and using the fact this linear functional is symmetric, i.e., the odd moments associated with this functional are zero, and also the fact that some hypergeometric representations associated with the Hermite polynomials are known. Observe that this can be done for other symmetric classical orthogonal polynomials. Moreover, we have obtained an integral representation for the generalized linear Hermite functional in the complex plane, and from this integral representation, we are able to obtain a novel integral representation for the Euler Gamma function.

Of course, this method can be applied not only to other (symmetric) classical orthogonal polynomials but to any other symmetric orthogonal polynomial sequence for which a hypergeometric representation is known for them. This is something we should do in order to obtain novel integral representations for other Special functions, for example we could consider some other generalization for the Hermite linear functional as well as some Laguerre-Hahn, or semi-classical, orthogonal polynomials (see e.g., [2,8] and the references therein).

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