2. Commission: Pure Mathematics

Sobolev OP: Connection formulae] Sobolev orthogonal polynomials: Connection formulae

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Abstract

The aim of this contribution is to obtain several connection formulae for the polynomial sequence which is orthogonal with respect to the discrete Sobolev inner product

$$\langle f,g\rangle_n = \langle \mathbf{u}, fg\rangle + \sum_{j=1}^{M} \mu_j f^{(\nu_j)}(c_j) g^{(\nu_j)}(c_j),$$

where $\mathbf{u}$ is a classical linear functional, $c_j \in \mathbb{R}$, $\nu_j \in \mathbb{N}_0$, $j = 1,2,\ldots.,M$. The Laguerre case will be considered.

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2.1 Introduction

In this paper we are going to consider sequences of polynomials orthogonal with respect to the discrete Sobolev inner product

$$\langle f,g\rangle = \langle \mathbf{u}, fg\rangle + \sum_{j=1}^{M} \mu_j f^{(\nu_j)}(c_j)g^{(\nu_j)}(c_j), \tag{2.1}$$

where $\mathbf{u}$ is a classical linear functional, $c_j, \mu_j \in \mathbb{C}$, and $\nu_j \in \mathbb{N}_0$, $j = 1,2,....,M$ in the wider sense possible for all the parameters and values related to the polynomials we want to study.

Observe that without loss of generality, we can assume that $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_M$. For this reason we will call the polynomials orthogonal with respect to (2.1) sequentially ordered Sobolev type orthogonal polynomials.

Observe one can express the inner product (2.1) in the following compact way [5]

$$\langle f,g\rangle = \langle \mathbf{u}, fg\rangle + (\mathbb{D}f)^T D\mathbb{D}g, \tag{2.2}$$

where $\mathbb{D}$ is the vector differential operator defined as

$$\mathbb{D}f := \left(f^{(\nu_1)}(x)\Big|_{x=c_1}, f^{(\nu_2)}(x)\Big|_{x=c_2}, \ldots, f^{(\nu_M)}(x)\Big|_{x=c_M} \right)^T,$$

D is the diagonal matrix with entries $\mu_1, \ldots, \mu_M$ and A^T is the transpose of the matrix A.

To simplify the notation we will write throughout the document $f^{(\nu)}(c)$ instead of $f^{(\nu)}(x)\Big|_{x=c}$.

Remark 2.1.1 Observe in the case when $\nu_1 = 0$, $\nu_2 = 1$, ..., $\nu_M = M-1$ and all the mass-points are equal each others the authors usually denote by $\mathbb{F}$ the matrix $\mathbb{D}f$ (see [6] and references therein).

For more detailed description of this Sobolev-type orthogonal polynomials (including the continuous ones) we refer the readers to the reviews [8, 9].

The structure of the paper is as follows: in Section 2.2 some preliminary results are quoted. In Section 2.3 all the algebraic results are presented such us several connection formulas for the sequentially ordered balanced Sobolev type orthogonal polynomials, an hypergeometric representation for these polynomials as well as some other algebraic relations between the classical orthogonal polynomials and the discrete-Sobolev ones.

2.2 Auxiliary results

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, and we use the sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the integers, real numbers and complex numbers respectively. Let $\mathbb{P}$ be the linear space of polynomials and let $\mathbb{P}'$ be its algebraic dual space.

We will also adopt the following notation: We denote by $\langle \mathbf{u}, p\rangle$ the duality bracket for $\mathbf{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$, and by $(\mathbf{u})_n = \langle \mathbf{u}, x^n\rangle$, with $n \geq 0$, the canonical moments of $\mathbf{u}$.

For any $n \in \mathbb{N}_0$, $a \in \mathbb{C}$, the Pochhammer symbol, or shifted factorial, is defined as

$$(a)_n := a(a+1)\cdots(a+n-1).$$

The Taylor polynomial of degree N is defined as

$$[f(x;c)]_N := \sum_{k=0}^{N} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

for every function f for which $f^{(k)}(c)$, $k = 0, 1, 2, ..., N$ exists.

The hypergeometric series is defined for $z \in \mathbb{C}$, $s, r \in \mathbb{N}_0$, $b_j \notin -\mathbb{N}$ as [7, §1.4]

$${}_rF_s\left(\begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_s)_k}{(b_1)_k \cdots (b_r)_k} \frac{x^k}{k!}.$$

Given a moment functional $\mathbf{u}$, it is said to be quasi-definite or regular (see [3]) if the Hankel matrix $H = \left((\mathbf{u})_{i+j}\right)_{i,j=0}^{\infty}$ associated with the moments of the functional is quasi-definite, i.e., all the n-by-n leading principal submatrices are regular for all $n \in \mathbb{N}_0$. Hence, there exists a sequence of polynomials $(P_n)_{n\geq 0}$ such that

1. The degree of P_n is n.
2. $\langle \mathbf{u}, P_n(x)P_m(x)\rangle = 0$, $m \neq n$.
3. $\langle \mathbf{u}, P_n^2(x)\rangle = d_n^2 \neq 0$, $n = 0, 1, 2, \dots$

Special cases of quasi-definite linear functionals are the classical ones (Jacobi, Laguerre, Hermite and Bessel).

We denote the n-th reproducing kernel by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{d_k^2}.$$

From the Christoffel-Darboux formula (see [3] or [13, Eq. (3.1)]), we have

$$K_n(x,y) = \frac{k_n}{k_{n+1}d_n^2} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y}, \tag{2.3}$$

where k_m is leading coefficient of $P_m(x)$, $m \in \mathbb{N}_0$.

We will use the following notation for the partial derivatives of $K_n(x,y)$:

$$\frac{\partial^{j+k}}{\partial x^j \partial y^k} K_n(x,y) = K_n^{(j,k)}(x,y), \quad 0 \le j,k \le n.$$

Note that, when $j = k = 0$, $K_n(x,y) = K^{(0,0)}(x,y)$ is the usual reproducing Kernel polynomial.

A direct consequence of the Christoffel-Darboux formula (2.3) is the following result [4, Eq. (16)]:

Proposition 2.2.1 The j-th partial derivative of the n-th reproducing kernel can be written as

$$K_n^{(j,0)}(x,y) = \frac{k_n\, j!}{k_{n+1}d_n^2} \frac{P_{n+1}(x)[P_n(x;y)]_j - P_n(x)[P_{n+1}(x;y)]_j}{(x-y)^j}. \tag{2.4}$$

Observe the following consequence, provided that c is not a zero of $P_n(x)$ for any n

$$\frac{\left|P_n^{(j)}(c)\right|^2}{d_n^2} = K_n^{(j,0)}(c,x) - K_{n-1}^{(j,0)}(c,x) \tag{2.5}$$

One last result will be useful to obtain some of our algebraic results.

Lemma 2.2.2 [1, Lemma 2.1] Let $M \in \mathbb{N}$, $\mathbf{u}$ be a classical linear form. Let $c_1, c_2, \dots, c_M \in \mathbb{R}$, $\nu_1, \nu_2, \dots, \nu_M \in \mathbb{N}_0$, and let us denote by $(S_n^{\vec{\mu}}(x;\vec{\nu},\vec{c}))$ the sequence of polynomials that are orthogonal with respect to the inner product (2.1). If c_i is not a zero of $S_n^{\vec{\mu}}(x;\vec{\nu},\vec{c})$, $i = 1,2,\dots,M$ for all $n \in \mathbb{N}_0$ then, there exists a polynomial, namely $\zeta(x)$, such that $\mathbb{D}\left(\zeta(x)S_n^{\vec{\mu}}(x;\vec{\nu},\vec{c})\right) = \vec{0}$ holds.

Remark 2.2.3 Observe that if all the c_i's are all different then $\zeta(x) = \prod_{j=1}^{M}(x-c_j)^{\nu_j+1}$, and if all of them are equal each others, i.e. $c_i = c$ for $i = 1,2,\dots,M$, then $\zeta(x) = (x-c)^{\nu_M+1}$.

Without loss of generality we denote by $\zeta(x)$ to the polynomial of minimum degree among all nonzero polynomials satisfying the conditions of the Lemma 2.2.2.

2.2.1 The Laguerre polynomials

Let $(L_n^\alpha(x))$ be the sequence of Laguerre polynomials, orthogonal with respect to the linear form $\mathbf{u}_\alpha$ on $\mathbb{P}$. These polynomial sequence is classical since $\mathbf{u}_\alpha$ fulfills the Pearson equation

$$\frac{d}{dx}[x\mathbf{x}] = (\alpha+1-x)\mathbf{x}.$$

Remark 2.2.4 Note that if $\Re(\alpha) > -1$ then, the linear from $\mathbf{u}_\alpha$ has the following integral representation (see for instance [12], [10, §18.3] or [7, §9.12]):

$$\langle \mathbf{u}_\alpha, f\rangle = \int_0^\infty f(x)x^\alpha e^{-x}dx,$$

and when $\alpha < 0$ the orthogonality of Sobolev-type is given in [12].

The Laguerre polynomial can be explicitly given in terms of hypergeometric series as

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}\; ;x\right). \tag{2.6}$$

Let us summarize some basic properties of the Laguerre orthogonal polynomials that will be used throughout this work.

Proposition 2.2.5 Let $(L_n^\alpha(x))$ be the Laguerre polynomials. The following statements hold:

- The three-term recurrence relation:

$$(n+1)L_{n+1}^\alpha(x) + (x-2n-\alpha-1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x), \quad n = 1,2,..., \tag{2.7}$$

  with initial conditions $L_0^\alpha(x) = 1$ and $L_1^\alpha(x) = \alpha+1-x$.

- The first structure relation:

$$x(L_n^\alpha(x))' = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x), \quad n = 1,2,... \tag{2.8}$$

- The second structure relation:

$$L_n^\alpha(x) = -\left(L_{n+1}^\alpha(x)\right)' + (L_n^\alpha(x))', \quad n = 0, 1, \dots \tag{2.9}$$

- The squared norm:

$$d_n^2 = \frac{(\alpha+1)_n}{n!}, \quad n = 0, 1, \dots \tag{2.10}$$

- The Ladder operators:

$$(L_n^\alpha(x))' = -L_{n-1}^{\alpha+1}(x), \quad n = 1, 2, \dots, \tag{2.11}$$

$$x(L_n^\alpha(x))' + (\alpha - x)L_n^\alpha(x) = (n+1)L_{n+1}^{\alpha-1}(x), \quad n = 0, 1, \dots \tag{2.12}$$

2.3 The connection formulae

A first step to get asymptotic properties is to obtain an adequate expression of the polynomials $L_n^{\alpha,\vec{\mu}}(x)$ in terms of the Laguerre polynomials, i.e., to solve the connection problem.

Remark 2.3.1

- Observe that, by construction, it is clear that $L_n^{\alpha,\vec{\mu}}(x) = L_n^\alpha(x)$ for $n = 0, 1, \dots, \nu_1 - 1$.
- For the part of algebraic calculations in this work the assumption of the dependency of the parameters μ_j of n is unnecessary, therefore we will omit such dependency.
- We assume that $L_n^{\alpha,\vec{\mu}}(x)$ has the same leading coefficient than $L_n^\alpha(x)$.

Since the Laguerre polynomials constitute a basis of the polynomials, we can consider the Fourier expansion of $L_n^{\alpha,\vec{\mu}}(x)$ in terms of such polynomial sequence.

Proposition 2.3.2 For every $n \geq \nu_1$ the following identity holds:

$$L_n^{\alpha,\vec{\mu}}(x) = L_n^\alpha(x) - \sum_{j=1}^{M} \mu_j \left(L_n^{\alpha,\vec{\mu}}(c_j)\right)^{(\nu_j)} K_{n-1}^{(0,\nu_j)}(x, c_j). \tag{2.13}$$

This is a classical result so the proof will be omitted. Still, we need to compute the values $\left(L_n^{\alpha,\vec{\mu}}(c_j)\right)^{(\nu_j)}$ for $j = 1, 2, \dots, M$ in order to have the complete expression. To do that we need to use an analogous result from [2, Proposition 2]. If we define $\mathbb{L}_n$ as $\mathbb{D}L_n^\alpha(x)$, $\mathbb{S}_n$ as $\mathbb{D}L_n^{\alpha,\vec{\mu}}(x)$ and $\mathbb{K}_n = \mathbb{D}_x^T \mathbb{D}_y K_n(x,y)$, then we need to solve the linear system

$$\mathbb{S}_n = \mathbb{L}_n - \mathbb{K}_{n-1}^T D^T \mathbb{S}_n.$$

Therefore, after some straightforward manipulations, (2.13) becames the desired compact connection formula [2, Proposition 2].

$$L_n^{\alpha,\vec{\mu}}(x) = L_n^{\alpha}(x) - \mathbb{L}_n^T(\mathbb{I}+D\mathbb{K}_{n-1})^{-1}D\mathbb{K}_{n-1}(x), \tag{2.14}$$

where $\mathbb{K}_n(x) = \mathbb{D}_y K_n(x,y)$.

This is an identity we expect to obtain, but there are other connection formulas.

Remark 2.3.3 Observe that the discrete Laguerre-Sobolev polynomials exits for all n if and only if the matrices $\mathbb{I}+D\mathbb{K}_{n-1}$ are regular for all $n = 1,2,\ldots$.

In the next result we establish a connection formula for the discrete Sobolev polynomials $L_n^{\alpha,\vec{\mu}}(x)$ similar to the one obtained in [11, Theorem 1] for non-varying discrete Sobolev orthogonal polynomials.

Let $\zeta(x)$ be the polynomial, of degree ν, we obtain from Lemma 2.2.2, then it is clear that for any two polynomials f and g, we have

$$\langle \zeta(x)f(x), g(x)\rangle = \langle \mathbf{u}_\alpha, \zeta(x)f(x)\overline{g(x)}\rangle = \langle f(x), \overline{\zeta(x)}g(x)\rangle. \tag{2.15}$$

Proposition 2.3.4 Let $(\zeta_j(x))_{j=0}^{\nu}$ be a sequence of polynomials, with $\zeta_\nu(x) = \zeta(x)$, such that $\deg \zeta_k(x) = k$ and it is a divisor of $\zeta_{k+1}(x)$ for $k = 0,1,\ldots,\nu-1$, and let $\left(P_n^{[\zeta_j^2]}(x)\right)$ be the polynomials orthogonal with respect to the linear functional $|\zeta_j(x)|^2\mathbf{u}_\alpha$, for $j = 0,1,\ldots,\nu$.

If the following conditions hold

$$P_n(c_j)P_{n-1}^{[\zeta_1^2]}(c_j)\cdots P_{n-\nu}^{[\zeta_\nu^2]}(c_j) \neq 0, \quad j = 1,2,\ldots,M, \tag{2.16}$$

then, there exists a family of coefficients $(\lambda_{j,n})_{j=0}^{\nu}$, not identically zero, such that for any $n \geq \nu$ the following connection formula holds:

$$L_n^{\alpha,\vec{\mu}}(x) = \sum_{j=0}^{\nu} \lambda_{j,n}\,\zeta_j(x)P_{n-j}^{[\zeta_j^2]}(x). \tag{2.17}$$

Another connection formula connects the discrete Sobolev polynomials with the the derivatives of the Laguerre polynomials which proof is similar to the previous result.

Proposition 2.3.5 For every $n \geq \nu$ the following identity holds:

$$L_n^{\alpha,\vec{\mu}}(x) = \sum_{k=0}^{\nu} \xi_{k,n} L_{n-k}^{\alpha+k}(x). \tag{2.18}$$

Observe that it is straightforward to extend these results to another classical families, even to a more generic frameworks such us discrete classical polynomials.

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