Orthogonality of the big -1 Jacobi polynomials for non-standard parameters

Howard S. Cohl and Roberto S. Costas-Santos

Abstract. The big -1 Jacobi polynomials $(Q_n^{(0)}(x;\alpha,\beta,c))_n$ have been classically defined for $\alpha,\beta\in(-1,\infty)$, $c\in(-1,1)$. We extend this family so that wider sets of parameters are allowed, i.e., they are non-standard. Assuming initial conditions $Q_0^{(0)}(x)=1$, $Q_{-1}^{(0)}(x)=0$, we consider the big -1 Jacobi polynomials as monic orthogonal polynomials which therefore satisfy the following three-term recurrence relation

$$xQ_n^{(0)}(x)=Q_{n+1}^{(0)}(x)+b_nQ_n^{(0)}(x)+u_nQ_{n-1}^{(0)}(x),\quad n=0,1,2,\ldots.$$

For standard parameters, the coefficients $u_n>0$ for all n. We discuss the situation where Favard's theorem cannot be directly applied for some positive integer n such that $u_n=0$. We express the big -1 Jacobi polynomials for non-standard parameters as a product of two polynomials. Using this factorization, we obtain a bilinear form with respect to which these polynomials are orthogonal.

1. Introduction

In 2011, Vinet and Zhedanov in [19] obtained a new family of "classical" orthogonal polynomials which can be obtained from the little q-Jacobi polynomials in the limit $q\to-1$. By "classical" it is meant that these polynomials satisfy (apart from a three-term recurrence relation) a nontrivial eigenvalue equation of the form

$$\mathcal{L}(p_n(x))=\lambda_np_n(x),$$

where $\mathcal{L}$ is a linear differential-difference operator which is of first order in the derivative ∂_x and contains a reflection operator R which acts as $Rf(x)=f(-x)$. They referred to these polynomials as being in the continuous -1 hypergeometric orthogonal polynomial scheme. This was illustrated by using a limiting process from the Askey–Wilson polynomials to the Bannai–Ito polynomials. In [20] the same authors applied this limit by starting with the big q-Jacobi polynomials in order to obtain hypergeometric representations for the big -1 Jacobi polynomials.

The polynomials the authors have considered in this -1 hypergeometric orthogonal polynomial scheme are q-polynomials for standard parameters, i.e., those values for the parameters whose weight function related to these families is positive definite (see [15]). The main aim of this work is to investigate the property of orthogonality for the big -1 polynomials for those values of the parameters, α, and β, for which the coefficient u_n in the recurrence relation (1.2) relative to these polynomials can be equal to zero for some $n\in\mathbb{N}_0$.

In the last three decades, some of the classical orthogonal polynomials with non-classical parameters have been provided with certain non-standard orthogonality properties. In the pioneering work by Kwon and Littlejohn in 1995 [9], Sobolev orthogonality for the Laguerre polynomials $L_n^{(-N)}(x)$ with $N\in\mathbb{N}$, was obtained using the following inner product:

$$\langle f,g\rangle=\sum_{m=0}^{N-1}\sum_{j=0}^{m}B_{m,j}(N)\left(f^{(m)}(0)g^{(j)}(0)+f^{(j)}(0)g^{(m)}(0)\right)+\int_0^\infty f^{(N)}(x)g^{(N)}(x)\,\mathrm{e}^{-x}\,\mathrm{d}x,$$

where the superscripts represent differentiation and

$$B_{m,j}(N):=\begin{cases}\displaystyle\sum_{p=0}^{j}(-1)^{m+j}\binom{N-1-p}{m-p}\binom{N-1-p}{j-p}, & 0\le j\le m\le N-1,\\[2ex] \displaystyle\frac12\sum_{p=0}^{m}\binom{N-1-p}{m-p}^2, & 0\le j=m\le N-1.\end{cases}$$

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In 1998, Álvarez de Morales and co-authors [3] found, using a different technique, the orthogonality for ultraspherical polynomials $C_n^{-N+\frac{1}{2}}$, where $N \in \mathbb{N}$. Later in [1], M. Alfaro et al. considered the cases for the Jacobi polynomials in which both parameters, α, and β, were negative integers, proving that in such a case the Jacobi polynomials satisfy a Sobolev orthogonality. In [2], M. Alfaro et al. considered the situation where the inner product can be written in the form

$$\langle f, g\rangle = F^t A G + \int f^{(N)}(x) g^{(N)}(x)\,\mathrm{d}\mu(x)\,, \tag{1.1}$$

where F and G are vectors obtained by evaluating f and g and maybe their derivatives at some points, A is a symmetric real matrix, and $\mathrm{d}\mu$ is the orthogonality measure associated with the Nth derivative of either the Laguerre, ultraspherical or Jacobi polynomials.

Note that for the parameters considered in this situation there exists a $n = N \in \mathbb{N}_0$ so that the coefficient u_n in the recurrence relation

$$x p_n = p_{n+1} + b_n p_n + u_n p_{n-1}\,, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

vanishes, i.e., $u_N = 0$. We say a non-standard parameter (or set of parameters) is degenerate if there exists at least one positive integer N for which $u_N = 0$.

The first term in the inner product (1.1) plays the role of the orthogonality for the set of polynomials of degree less than N, since if f, g are two polynomials, with $\deg f$, $\deg g < N$, then $f^{(N)}(x) = g^{(N)}(x) = 0$, and one has

$$\langle f, g\rangle = F^t A G.$$

In this term, the points for the evaluation are the roots of p_N; this ensures that the inner product vanishes when one entry is p_n with $n \geq N$, since p_N is a factor of any p_n with $n \geq N$. The second term is relevant for polynomials with degree greater than N, and it is needed in order to have an orthogonality characterizing the sequence $(p_n)_{n=0}^{\infty}$. The technique used in [3] is applicable for all classical orthogonal polynomials where u_n vanishes at certain $n = N \in \mathbb{N}_0$, and in fact it has been used, among others, in [11, 12, 13, 14].

In 2009 [5], Costas-Santos and Sanchez-Lara studied this problem for classical discrete polynomials, i.e., discrete polynomials in the Askey Scheme of generalized hypergeometric orthogonal polynomials. We consider classical discrete polynomials for which there exists N for which $u_N = 0$, i.e., for non-standard degenerate parameters. Note that this can only happen for the Racah, Hahn, dual Hahn, and Krawtchouk polynomials which are the Wilson, continuous Hahn, continuous dual Hahn and Meixner polynomials for which some of its parameters are equal to a negative integer. The corresponding three-term recurrence relation for these polynomials also presents one vanishing coefficient i.e., $u_N = 0$, and the inner product found can be written as

$$\langle f, g\rangle = \int f(x) g(x)\,\mathrm{d}\mu_d(x) + \int f^{(N)}(x) g^{(N)}(x)\,\mathrm{d}\mu_c(x)\,, \tag{1.3}$$

where $\mathrm{d}\mu_d$ is a discrete measure with a finite number of masses and $\mathrm{d}\mu_c$ is an absolutely continuous measure. For more details and further reading see [5, 6, 8, 16, 18] and references therein. The organization of the paper is as follows. In Section 2 we provide some preliminary material and in Section 3 we obtain the property of orthogonality for the big -1 Jacobi polynomials for non-standard degenerate parameters.

2. Preliminary material and notations

We adopt the following notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, and we use the sets $\mathbb{P}$, $\mathbb{P}'$, to represent the linear space of polynomials with complex coefficients and its algebraic dual space. We denote by $\langle \mathbf{u}, p\rangle$ the duality bracket for $\mathbf{u} \in \mathbb{P}'$ and $p \in \mathbb{P}$. The big -1 Jacobi polynomials have representations given in terms of the Gauss hypergeometric function defined as [17, (15.2.1)]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.1}$$

where $|z| < 1$ and the shifted factorial is defined as [17, (5.2.4-5)] $(a)_n := (a)(a+1)\cdots(a+n-1)$, $a \in \mathbb{C}$, $n \in \mathbb{N}_0$. The following ratio of two gamma functions [17, Chapter 5] are related to the shifted factorial, namely for $a \in \mathbb{C} \setminus -\mathbb{N}_0$, one has

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.2}$$

which allows one to extend the definition to non-positive integer values of n. For a description of the properties of the gamma function, see [17, hrefhttp://dlmf.nist.gov/5Chapter 5].

3. Big −1 Jacobi polynomials

The monic big −1 Jacobi polynomials can be defined in terms of the Gauss hypergeometric function as follows:

(3.1)
$$Q_n^{(0)}(x) := \kappa_n \begin{cases} {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+1}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) + \dfrac{n(1-x)}{(c+1)(\alpha+1)}\, {}_2F_1\left(\begin{matrix} 1-\frac{n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{\alpha+3}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right), & \text{if } n \text{ even}, \\ {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+1}{2} \\ \frac{\alpha+1}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right) - \dfrac{(\alpha+\beta+n+1)(1-x)}{(1+c)(\alpha+1)}\, {}_2F_1\left(\begin{matrix} -\frac{n-1}{2}, \frac{n+\alpha+\beta+3}{2} \\ \frac{\alpha+3}{2} \end{matrix}; \frac{1-x^2}{1-c^2}\right), & \text{if } n \text{ odd}, \end{cases}$$

where κ_n is defined as

$$\kappa_n := \begin{cases} \dfrac{(1-c^2)^{\frac{1}{2}n}(\frac{1}{2}(\alpha+1))_{\frac{1}{2}n}}{(\frac{1}{2}(n+\alpha+\beta+2))_{\frac{1}{2}n}}, & \text{if } n \text{ even}, \\ (c+1)\dfrac{(1-c^2)^{\frac{1}{2}(n-1)}(\frac{1}{2}(\alpha+1))_{\frac{1}{2}(n-1)}}{(\frac{1}{2}(n+\alpha+\beta+1))_{\frac{1}{2}(n-1)}}, & \text{if } n \text{ odd}. \end{cases} \tag{3.2}$$

The big −1 Jacobi polynomials $Q_n^{(0)}(x;\alpha,\beta,c)$, satisfy the three-term recurrence relation [20, (2.20), (2.21)]:

$$xQ_n^{(0)}(x) = Q_{n+1}^{(0)}(x) + b_n Q_n^{(0)}(x) + u_n Q_{n-1}^{(0)}(x), \tag{3.3}$$

where

$$b_n := \begin{cases} -c + \dfrac{(c-1)n}{\alpha+\beta+2n} + \dfrac{(c+1)(\beta+n+1)}{\alpha+\beta+2n+2}, & \text{if } n \text{ even}, \\ c - \dfrac{(c-1)(n+1)}{\alpha+\beta+2n+2} - \dfrac{(c+1)(\beta+n)}{\alpha+\beta+2n}, & \text{if } n \text{ odd}, \end{cases} \tag{3.4}$$

and

$$u_n := \begin{cases} \dfrac{(c-1)^2 n(\alpha+\beta+n)}{(\alpha+\beta+2n)^2}, & \text{if } n \text{ even}, \\ \dfrac{(c+1)^2(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}, & \text{if } n \text{ odd}, \end{cases} \tag{3.5}$$

with initial conditions $Q_{-1}^{(0)}(x) = 0$, $Q_0^{(0)}(x) = 1$. For any real $c \neq 1$ and real α, β satisfying the restriction $\alpha, \beta > -1$, the recurrence coefficients b_n, u_n are real and positive. Hence the big −1 Jacobi polynomials are positive definite orthogonal polynomials.

Remark 3.1. Observe that if α (resp. β) is a negative odd integer, namely $\alpha = -2N+1$ (resp. $\beta = -2N+1$) then $u_{2N-1} = 0$, i.e., α (resp. β) is non-standard degenerate parameter. So we can apply the degenerate version of Favard's theorem (see [6, Theorem 2.2]), i.e., there exist moment functionals, $\mathscr{L}_0$ and $\mathscr{L}_N$, such that the polynomial sequence is orthogonal with respect to the bilinear form

$$\langle p, r\rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{T}^{(2N-1)}p\mathscr{T}^{(2N-1)}r), \quad p, r \in \mathbb{P},$$

where $\mathscr{T}$ is a certain lowering operator.

Taking the former remark into account, the Gauss hypergeometric function with a suitable normalization factorizes as follows.

Lemma 3.2 (Factorization). *Let $n, N \in \mathbb{N}$, $a \in \mathbb{C}$. Then*

$$(-N+1)_{n+N}\, {}_2F_1\left(\begin{matrix} -n-N, a \\ -N+1 \end{matrix}; x\right) = (N+1)_n(-N+1)_N\, {}_2F_1\left(\begin{matrix} -N, a \\ -N+1 \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} -n, a+N \\ N+1 \end{matrix}; x\right). \tag{3.6}$$

Proof. By definition one has

$$\begin{aligned}(-N+1)_{n+N}\, {}_2F_1\left(\begin{matrix} -n-N, a \\ -N+1 \end{matrix}; x\right) &= (-N+1)_{n+N} \sum_{k=0}^{n+N} \frac{(-n-N, a)_k}{(-N+1, 1)_k x^k} \\ &= \sum_{k=0}^{n+N} \frac{(-n-N, a)_k}{(1)_k}(-N+k+1)_{n+N-k} x^k.\end{aligned}$$

Then setting $k = k + N$ we have

$$(-N+1)_{n+N}\, {}_2F_1\left(\begin{matrix}-n-N, a\\ -N+1\end{matrix}; x\right) = \sum_{k=0}^{n} \frac{(-n-N,a)_{k+N}}{(1)_{k+N}}(k+1)_{n-k}x^{k+N}$$
$$= \frac{(-n-N,a)_N n!}{N!}x^N \sum_{k=0}^{n}\frac{(-n,a+N)_k}{(N+1,1)_k}x^k$$
$$= (N+1)_n(a)_N(-x)^N\, {}_2F_1\left(\begin{matrix}-n, a+N\\ N+1\end{matrix}; x\right). \tag{3.7}$$

Setting $n = 0$ in (3.7), one obtains

$$(-N+1)_N\, {}_2F_1\left(\begin{matrix}-N, a\\ -N+1\end{matrix}; x\right) = (a)_N(-x)^N. \tag{3.8}$$

Combining (3.7) and (3.8) produces the result. $\square$

Now that we have obtained factorization for the Gauss hypergeometric function, we will use it to obtain a factorization for the big -1 Jacobi polynomials when the parameter α (resp. β) is equal to $-2N-1$ for some positive integer N.

Lemma 3.3 (Factorization of big -1 Jacobi polynomials). *Let $m, M, N \in \mathbb{N}_0$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$, with $\alpha = -2N-1$ or $\beta = -2N-1$. Then*

$$Q^{(0)}_{2N+1}(x; -2N-1, \beta, c) = (x^2-1)^N(x-1), \tag{3.9}$$
$$Q^{(0)}_{2N+1}(x; \alpha, -2N-1, c) = (x^2-c^2)^N(x+c), \tag{3.10}$$
$$Q^{(0)}_{2N+1+m}(x; -2N-1, \beta, c) = (-1)^m(x^2-1)^N(x-1)Q^{(0)}_m(-x; 2N+1, \beta, -c), \tag{3.11}$$
$$Q^{(0)}_{2N+1+m}(x; \alpha, -2N-1, c) = (x^2-c^2)^N(x+c)Q^{(0)}_m(x; \alpha, 2N+1, -c). \tag{3.12}$$

Proof. Identities (3.9) and (3.10) follow from (3.12) and (3.11) by setting $m = 0$ in these. We will first prove (3.11) and then (3.12). Consider $m = 2M$, $n = 2N+1+m$. Then one has

$$Q^{(0)}_{2N+1+m}(x; \alpha, -2N-1, c) = \kappa_n\left({}_2F_1\left(\begin{matrix}-N-M, M+\widehat{\beta}\\ -N\end{matrix}; \frac{1-x^2}{1-c^2}\right)\right.$$
$$\left. + \frac{(M+\widehat{\beta})(1-x)}{(c+1)N}\, {}_2F_1\left(\begin{matrix}-N-M, M+1+\widehat{\beta}\\ -N+1\end{matrix}; \frac{1-x^2}{1-c^2}\right)\right), \tag{3.13}$$

where $\widehat{\beta} = (\beta+1)/2$. Next using Lemma 3.2 and after a tedious calculation comparing the final expression with the one for $Q_m(-x; 2N+1, \beta, -c)$ and its corresponding normalization coefficient, the identity for the even m values holds. For $n = 2N+1+2M+1$, i.e., $m = 2M+1$, for $\alpha = -2N-1$, the proof is similar and will be omitted. Hence (3.11) follows. The identity (3.12) holds by applying the same procedure, which completes the proof. $\square$

A different way to guess and then obtain the factorization for the big -1 Jacobi polynomials for non-standard parameters throughout is to use the following two step procedure. The first step is to use Lemma 3.2 in order to prove the identities (3.9), (3.10). Once we have that, it is straightforward to find, e.g., for $m \geq 0$, one has

$$Q^{(0)}_{2N+1+m}(x; -2N-1, \beta, c) = Q^{(0)}_{2N+1}(x; -2N-1, \beta, c)P_m(x),$$

where $P_m \in \mathbb{P}$ of degree m. In the second step, we obtain a new recurrence relation for these new polynomials and comparing the recurrence coefficients, we prove this new polynomial sequence is related to the original one.

Lemma 3.4. *For any $n, N \in \mathbb{N}$, $\alpha, \beta, c \in \mathbb{C}$, $c \neq \pm 1$, with $\alpha = -2N-1$ (resp. $\beta = -2N-1$), the following recurrence relation holds for $n \geq 0$:*

$$xQ^{(0)}_n(x; 2N+1, \beta, -c) = -Q^{(0)}_{n+1}(x; 2N+1, \beta, -c) + \widehat{b}_n Q^{(0)}_n(x; 2N+1, \beta, -c) + \widehat{u}_n Q^{(0)}_{n-1}(x; 2N+1, \beta, -c),$$

where

$$\widehat{b}_n = -b_{n+2N+1}(-2N-1, \beta, c) = b_n(2N+1, \beta, -c), \qquad \widehat{u}_n = u_{n+2N+1}(-2N-1, \beta, c) = u_n(2N+1, \beta, -c).$$

Respectively, the following recurrence relation holds for $n \geq 0$:

$$xQ^{(0)}_n(x; \alpha, 2N+1, -c) = Q^{(0)}_{n+1}(x; \alpha, 2N+1, -c) + \widetilde{b}_n Q^{(0)}_n(x; \alpha, 2N+1, -c) + \widetilde{u}_n Q^{(0)}_{n-1}(x; \alpha, 2N+1, -c),$$

where

$$\widetilde{b}_n = b_{n+2N+1}(\alpha,-2N-1,c) = b_n(\alpha,2N+1,-c), \qquad \widetilde{u}_n = u_{n+2N+1}(\alpha,-2N-1,c) = u_n(\alpha,2N+1,-c).$$

PROOF. We start by considering $\alpha=-2N-1$, then one has $u_{2N+1}=0$. In fact,

$$xQ^{(0)}_{2N+1}(x;2N+1,\beta,c) = (x^2-1)^N(x-1).$$

If we consider $n\geq m+2N+1$, then

$$Q^{(0)}_{2N+1+m}(x;-2N-1,\beta,c) = Q^{(0)}_{2N+1}(x;-2N-1,\beta,c)(-1)^m Q^{(0)}_m(-x;2N+1,\beta,-c),$$

and therefore the recurrence relation for these polynomials can be written as

$$\begin{aligned} x(-1)^m Q^{(0)}_m(-x;2N+1,\beta,-c) =& (-1)^{m+1}Q^{(0)}_{m+1}(-x;2N+1,\beta,-c) + b_{m+2N+1}(-1)^m Q^{(0)}_m(-x;2N+1,\beta,-c) \\ &+ u_{m+2N+1}(-1)^{m-1}Q^{(0)}_{m-1}(-x;2N+1,\beta,-c). \end{aligned}$$

After some straightforward calculations the result follows. For $\beta=-2N-1$, the derivation is analogous which completes the proof. $\square$

Note that if $u_{2N+1}=0$ for some $N\in\mathbb{N}$ then the orthogonality property for the big -1 Jacobi polynomials holds for polynomials of degree less or equal to $2N+1$. In the next result, we construct, thanks to the factorization of the big -1 polynomials, a new property of orthogonality that is valid for the big -1 Jacobi polynomials even when $n>2N+1$, i.e., we can extend the property of orthogonality for the big -1 polynomials for non-standard degenerate parameters.

THEOREM 3.5 (Orthogonality of big -1 Jacobi polynomials for non-standard parameters). *Let $N\in\mathbb{N}_0$, $c\in\mathbb{C}$, $c\neq\pm1$. Define the linear operator* $\mathbf{u}$ [20, Section 4] *as*

$$\langle \mathbf{u}, pq\rangle := \int_{[-c,-1]\cup[1,c]} p(x)q(x)\,\frac{x}{|x|}(x+1)(c-x)(x^2-1)^{\frac12(\alpha-1)}(c^2-x^2)^{\frac12(\beta-1)}\,\mathrm{d}x.$$

The norm with respect to the linear functional $\mathbf{u}\in\mathbb{P}'$ is given by $h_n := h_n(\alpha,\beta,c)$,

$$h_n(\alpha,\beta,c) := \langle \mathbf{u}, Q^{(0)}_n Q^{(0)}_n\rangle = \langle \mathbf{u},1\rangle \begin{cases} \dfrac{2(c^2-1)^n(\frac12 n)!(\frac12(\alpha+1),\frac12(\beta+1))_{\frac12 n}}{(\frac12(\alpha+\beta)+1)_n(\frac12(\alpha+\beta+n)+1)_{\frac12 n}}, & \text{if } n \text{ even},\\[2ex] \dfrac{2(c-1)^{n-1}(c+1)^{n+1}(\frac12(n-1))!(\frac12(\alpha+1),\frac12(\beta+1))_{\frac12(n+1)}}{(\frac12(\alpha+\beta)+1)_n(\frac12(\alpha+\beta+n+1))_{\frac12(n+1)}}, & \text{if } n \text{ odd}. \end{cases} \tag{3.14}$$

Then, the polynomial sequences

$$(Q^{(0)}_n(x;-2N-1,\beta,c)), \quad (Q^{(0)}_n(x;\alpha,-2N-1,c)), \tag{3.15}$$

are orthogonal with respect to the bilinear forms

$$\langle p,q\rangle_1 = \mathscr{L}_0(p,q) + \lambda_N(\beta)\langle \mathbf{u}_1, (\tau_\alpha^{2N+1}p)(\tau_\alpha^{2N+1}q)\rangle, \tag{3.16}$$

$$\langle p,q\rangle_2 = \mathscr{L}_1(p,q) + \lambda_N(\alpha)\langle \mathbf{u}_2, (\tau_\beta^{2N+1}p)(\tau_\beta^{2N+1}q)\rangle, \tag{3.17}$$

respectively. Here τ_α, τ_β are linear operators defined such that

$$\tau_\alpha[Q^{(0)}_n(x;\alpha,\beta,c)] = Q_{n-1}(-x;\alpha+2,\beta,-c),$$

$$\tau_\beta[Q^{(0)}_n(x;\alpha,\beta,c)] = Q_{n-1}(x;\alpha,\beta+2,-c),$$

and

$$\langle \mathbf{u}_1, f(x,\alpha,\beta,c)\rangle := \langle \mathbf{u}, f(-x,-\alpha,\beta,-c)\rangle, \quad \langle \mathbf{u}_2, f(x,\alpha,\beta,c)\rangle := \langle \mathbf{u}, f(x,\alpha,-\beta,-c)\rangle,$$

$\mathscr{L}_0$ and $\mathscr{L}_1$ are two moment linear functionals, and

$$\lambda_N(\alpha) = \frac12 h_{2N+1}(-2N-1,\alpha,c), \quad \lambda_N(\beta) = \frac12 h_{2N+1}(-2N-1,\beta,c).$$

PROOF. Let us consider the situation corresponding to (3.16), i.e., $\alpha=-2N-1$, with $N\in\mathbb{N}$, $\beta\in\mathbb{C}$. We need to prove the the properties of orthogonality $\langle Q^{(0)}_n, Q^{(0)}_m\rangle_1 = 0$ for all $m<n$, $\langle Q^{(0)}_n, Q^{(0)}_n\rangle_1 \neq 0$ for all n in order to prove that $(Q^{(0)}_n)$ is a monic orthogonal polynomial sequence with respect to $\langle,\rangle_1$. Let us prove this in three steps: (i) $m\leq n<2N+1$; (ii) $m<2N+1\leq n$; (iii) $2N+1\leq n\leq m$. (i): If $m\leq n<2N+1$ the coefficient u_k of the recurrence relation is non-zero for $k=0,1,\dots,2N$, so by the Favard result (see [4]), there exists a moment linear functional, namely $\mathscr{L}_0$ such that the big -1 Jacobi polynomials $(Q_j^{(0)})_{j=0}^{2N}$ are orthogonal with respect to it. Due to the fact that $\tau_\alpha^{2N+1}(Q_j) = 0$ for $j = 0, 1, ..., 2N$, therefore the property of orthogonality holds in this case, i.e.,

$$\langle Q_n^{(0)}, Q_m^{(0)} \rangle_1 = \langle \mathbf{u}, Q_n^{(0)} Q_m^{(0)} \rangle = h_n(-2N-1, \beta, c)\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker (delta) symbol. (ii): If $m < 2N+1 \le n$, then $\mathscr{L}_0(Q_n^{(0)}, Q_m^{(0)}) = 0$; at the same time $\tau_\alpha^{2N+1}(Q_j) = 0$ for $j = 0, 1, ..., 2N$, therefore the property of orthogonality holds in this case. (iii): If $2N+1 \le n \le m$, then $\mathscr{L}_0(Q_n^{(0)}, Q_m^{(0)}) = 0$; and since

$$\tau_\alpha^{2N+1} Q_n^{(0)}(x; -2N-1, \beta, c) = Q_{n-(2N+1)}^{(0)}(-x; 2N+1, \beta, -c),$$

is orthogonal with respect to the linear functional $\mathbf{u}$, one has

$$\begin{aligned}\langle Q_n^{(0)}, Q_m^{(0)} \rangle_1 =& \lambda_N(\beta) \langle \mathbf{u}, \tau_\alpha^{2N+1} Q_n^{(0)}(x; -2N-1, \beta, c) \tau_\alpha^{2N+1} Q_m^{(0)}(x; -2N-1, \beta, c) \rangle \\ =& \lambda_N(\beta) \langle \mathbf{u}, Q_{n-2N-1}^{(0)}(-x; 2N+1, \beta, -c) Q_{m-2N-1}^{(0)}(-x; 2N+1, \beta, -c) = h_n(-2N-1, \beta, c)\delta_{n,m},\end{aligned}$$

and the property of orthogonality holds in this situation. Moreover, since the polynomials are monic, it is straightforward to check using the recurrence relation (3.3) that

$$h_n = \langle \mathbf{u}, x Q_n^{(0)}(x) Q_{n-1}^{(0)}(x) \rangle = u_n \langle \mathbf{u}, x Q_{n-1}^{(0)}(x) Q_{n-1}^{(0)}(x) \rangle = u_n h_{n-1}. \tag{3.18}$$

Furthermore, the norm squared (3.14) and the recurrence coefficient u_n given in (3.3) satisfies (3.18). The situation which corresponds to (3.17), i.e., $\beta = -2N-1$, with $N \in \mathbb{N}$, $\alpha \in \mathbb{C}$ is similar so we will omit its proof. Hence the result holds. □

Remark 3.6. Note that if $\alpha = -2N-1$ and $\beta \in \mathbb{C}$ and due to (3.9) one must consider the moment linear form [10]

$$\mathscr{L}_0(p, q) = \sum_{j=0}^{N} \lambda_j \, p^{(j)}(1)\, q^{(j)}(1) + \sum_{j=0}^{N-1} \mu_j \, p^{(j)}(-1)\, q^{(j)}(-1),$$

where λ_j, μ_j are positive. So, taking into account (3.11), we have

$$\mathscr{L}_0(Q_j^{(0)}(x), \pi(x)) = 0, \quad j \ge 2N+1,$$

for any $\pi \in \mathbb{P}$. Moreover, by construction, if $0 \le j < 2N+1$ one has

$$\langle Q_j^{(0)}(x), Q_j^{(0)}(x) \rangle_1 = \mathscr{L}_0 \left(Q_j^{(0)}(x) p, Q_j^{(0)}(x) \right) = \langle \mathbf{u}, Q_j^{(0)} Q_j^{(0)} \rangle = h_j \neq 0.$$

If $j \ge 2N+1$ and taking into account Lemma 3.4 one has

$$\begin{aligned}\langle Q_j^{(0)}(x), Q_j^{(0)}(x) \rangle_1 =& \lambda_N(\beta) \langle \mathbf{u}, Q_{j-2N-1}^{(0)}(-x; 2N+1, \beta, -c) Q_{j-2N-1}^{(0)}(-x; 2N+1, \beta, -c) \rangle \\ =& \lambda_N(\beta) h_{j-2N-1}(2N+1, \beta, -c) = h_j(-2N-1, \beta, c).\end{aligned}$$

Observe the last identity must be considered as the formal limit:

$$\tfrac{1}{2} \lim_{\epsilon \to 0} h_{2N+1}(-2N-1+\epsilon, \beta, c) h_{j-2N-1}(2N+1+\epsilon, \beta, -c) = \lim_{\epsilon \to 0} h_j(-2N-1+\epsilon, \beta, c).$$

For $\beta = -2N-1$, it is analogous since in (3.10) one must consider the moment linear form [10]

$$\mathscr{L}_1(p, q) = \sum_{j=0}^{N} \kappa_j p^{(j)}(1) q^{(j)}(-c) + \sum_{j=0}^{N-1} \ell_j p^{(j)}(-1) q^{(j)}(c),$$

where κ_j, ℓ_j are positive. Moreover, in a similar way

$$\frac{1}{2} \lim_{\epsilon \to 0} h_{2N+1}(\alpha, -2N-1+\epsilon, c) h_{j-2N-1}(\alpha, 2N+1+\epsilon, -c) = \lim_{\epsilon \to 0} h_j(\alpha, -2N-1+\epsilon, c).$$

4. Conclusion and future work

We have studied the big -1 Jacobi polynomials, which are a $q \to -1$ limit of the Bannai–Ito polynomials, when at least one of the parameters α, β, are non-standard and the coefficient u_n of the three-term recurrence relation (3.3) is equal to zero for a certain index, namely N, i.e., $\alpha = -2N - 1$ or $\beta = -2N - 1$ where N is a positive integer. We have also obtained the Gauss hypergeometric representation, its factorization, and the property of orthogonality using similar techniques that were used in [5].

In [6, Section 4], a similar procedure for the big q-Jacobi polynomials was applied for non-standard parameters for which the u_n coefficient of its recurrence relation is equal to zero for some $n \in \mathbb{N}_0$. Along these lines we are investigating the method for obtaining the orthogonality property of the big -1 Jacobi polynomials for non-standard parameters following a procedure analogous to the one followed by T. E. Pérez and M. A. Piñar in [16] for the generalized Laguerre polynomials. In [6] the authors also studied the q-Racah polynomials for non-standard parameters for which the u_n coefficient vanishes for some $n \in \mathbb{N}_0$ and since the Bannai–Ito polynomials can be obtained from the q-Racah polynomials ([7, §14.2]) by taking an analogous limit $q \to -1$ (see [20]), it makes sense to perform a similar study for the Bannai–Ito polynomials. We are also working on obtaining explicit hypergeometric representations, the factorization, and the property of orthogonality for the Bannai–Ito polynomials and other families of the continuous -1 hypergeometric orthogonal polynomial scheme for non-standard parameters.

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Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, USA

Email address: howard.cohl@nist.gov

Department of Quantitative Methods, Universidad Loyola Andalucía, Sevilla, Spain

Email address: rscosa@gmail.com