Special values for the continuous q-Jacobi polynomials with application to its Poisson kernel

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Abstract. We study special values for the continuous q-Jacobi polynomials and present applications of these special values, which arise from bilinear generating functions and in particular, the Poisson kernel for these polynomials.

1 Preliminaries

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, and we use the sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the integers, real numbers, and complex numbers respectively, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{C}^\dagger := \mathbb{C}^* \setminus \{z \in \mathbb{C} : |z| = 1\}$. We also adopt the following *multiset* notation and conventions. Let $\mathbf{a} := \{a, b, c, d\}$, $a, b, c, d \in \mathbb{C}^*$. Throughout the paper, we assume that the empty sum vanishes and the empty product is unity.

Definition 1.1. *We adopt the following conventions for succinctly writing elements of lists. To indicate sequential positive and negative elements, we write*

$$\pm a := \{a, -a\}.$$

We also adopt the analogous notations

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\}, \quad z^{\pm} := \{z, z^{-1}\}.$$

Within a list of items, we define

$$a + \left\{ \begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix} \right\} := \{a + x_1, \ldots, a + x_n\}.$$

In the same vein, consider the numbers $f_s \in \mathbb{C}$ with $s \in \mathcal{S} \subset \mathbb{N}$, with $\mathcal{S}$ finite. Then, the notation $\{f_s\}$ represents the multiset of all complex numbers f_s such that $s \in \mathcal{S}$. Furthermore, consider some $p \in \mathcal{S}$, then the notation $\{f_s\}_{s\neq p}$ represents the sequence of all complex numbers f_s such that $s \in \mathcal{S}\setminus\{p\}$. In addition, for the empty list, $n = 0$, we take

$$\{a_1, ..., a_n\} := \emptyset.$$

Define the set $\Omega_q := \{q^{-k} : k \in \mathbb{N}_0\}$. In this paper we will be using standard notations for both finite and q-shifted factorials: including multi-q-shifted factorial notation where a comma delineated list such as $(a_1,\ldots,a_r;q)_k$ represents the products $(a_1;q)_k\cdots(a_r;q)_k$ (see [5, Appendix I]). We also adopt standard notations for both terminating and nonterminating basic hypergeometric functions ${}_r\phi_s$ [10, (17.4.1)]

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right) := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{1} $$

where $b_1,\ldots,b_s\not\in\Omega_q$. For $s+1>r$, ${}_r\phi_s$ is an entire function of z, for $s+1=r$ then ${}_r\phi_s$ is convergent for $|z|<1$, and for $s+1<r$ the series is divergent unless it is terminating. We will also refer to very-well-poised basic hypergeometric functions ${}_{r+1}W_r$ [5, (2.1.11)]

$$ {}_{r+1}W_r(a;a_4,\ldots,a_{r+1};q,z) := {}_{r+1}\phi_r\left(\begin{matrix} \pm q\sqrt{a},a,a_4,\ldots,a_{r+1} \\ \pm\sqrt{a},\frac{qa}{a_4},\ldots,\frac{qa}{a_{r+1}} \end{matrix};q,z\right), \tag{2} $$

where $\sqrt{a},\frac{qa}{a_4},\ldots,\frac{qa}{a_{r+1}}\not\in\Omega_q$. For those not familiar with these functions, we urge the reader to refer to some important standard references on the subject, and in particular [10, Chapter 17] and links therein and the entire book by Gasper & Rahman (2004) [5]. For particulars relating to orthogonal polynomials we suggest the reader to refer to [7, Chapters 1,9,14], the monograph [6] and the Memoirs of AMS article by Askey & Wilson [2]. For a general treatment of special functions, one should refer to [1] and [10].

1.1 The Askey–Wilson polynomials

The Askey–Wilson polynomials can be defined in terms of the terminating balanced basic hypergeometric series [7, (14.1.1)]

$$ p_n(x;\mathbf{a}|q) := a^{-n}(ab,ac,ad;q)_n\,{}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}abcd,az^{\pm} \\ ab,ac,ad \end{matrix};q,q\right), \tag{3} $$

where $x=\frac12(z+z^{-1})$. In some of the derivations given below, we use the renormalized version of the Askey–Wilson polynomials given by

$$ r_n(x;\mathbf{a}|q) := {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}abcd,az^{\pm} \\ ab,ac,ad \end{matrix};q,q\right) = \frac{a^n}{(ab,ac,ad;q)_n}\,p_n(x;\mathbf{a}|q). \tag{4} $$

The Askey–Wilson polynomials have the following special values [8, (114)]

$$ p_n(\tfrac12(a+a^{-1});a,b,c,d|q) = a^{-n}(ab,ac,ad;q)_n \tag{5} $$

(and similarly for arguments $\frac12(b+b^{-1})$, $\frac12(c+c^{-1})$, $\frac12(d+d^{-1})$).

1.2 The q-Racah polynomials

Let $m\in\mathbb{C}$, $n,N\in\mathbb{N}_0$ such that $n\in\{0,\ldots,N\}$. Let us consider the q-Racah polynomials $R_n(\mu(m);\bar\alpha,\bar\beta,\bar\gamma,\bar\delta|q)$ which are discrete cases of the Askey–Wilson polynomials (3) (note that we have used the notation $\{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta\}$ instead of the standard $\{\alpha,\beta,\gamma,\delta\}$ in order to disambiguate between the $\{\alpha,\beta\}$ parameters which appear in the study of continuous q-Jacobi polynomials (see §1.3 below)). The q-Racah polynomials are defined by [7, (14.2.1)]

$$ R_n(\mu(m);\bar\alpha,\bar\beta,\bar\gamma,\bar\delta|q) := {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n+1}\bar\alpha\bar\beta,q^{-m},q^{m+1}\bar\gamma\bar\delta \\ q\bar\alpha,q\bar\beta\bar\delta,q\bar\gamma \end{matrix};q,q\right), \tag{6} $$

where $\mu(m):=\mu(m;\bar\gamma,\bar\delta|q):=q^{-m}+q^{m+1}\bar\gamma\bar\delta$, and

$$ \bar\alpha=q^{-N-1}\quad\text{or}\quad\bar\beta\bar\delta=q^{-N-1}\quad\text{or}\quad\bar\gamma=q^{-N-1}. \tag{7} $$

The q-Racah polynomials are orthogonal on the finite q-quadratic set $\{\mu(m;\bar{\gamma},\bar{\delta}|q)\}$. See [7, (14.2.2)] for the orthogonality relation for q-Racah polynomials. Observe that for $m\in\mathbb{N}_0$, the q-Racah polynomials satisfy the following duality relation [8, (146)]

$$R_n(\mu(m);\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}|q)=R_m(\mu(n);\bar{\gamma},\bar{\delta},\bar{\alpha},\bar{\beta}|q). \tag{8}$$

1.3 The continuous q-Jacobi polynomials

The continuous q-Jacobi polynomials [7, Section 14.10] are particular cases of the Askey–Wilson polynomials as follows [7, (14.1.19) and p. 467]

$$P_n^{(\alpha,\beta)}(x|q):=\frac{q^{(\frac{\alpha}{2}+\frac14)n}}{(q,-q^{\frac{\alpha+\beta}{2}+\frac12\{\substack{1\\2}\}};q)_n}p_n(x;\mathbf{a}|q), \tag{9}$$

where

$$\mathbf{a}:=\{a,b,c,d\}:=\left\{q^{\frac12\alpha+\frac14\{\substack{1\\3}\}},-q^{\frac12\beta+\frac14\{\substack{1\\3}\}}\right\}. \tag{10}$$

For the coefficients $\{a,b\}=\{\pm q^{\frac12}\}$, $c=q^{\alpha+\frac12}$, $d=-q^{\beta+\frac12}$, the renormalized Askey–Wilson polynomials are related to the continuous q-Jacobi polynomials with base q^2, namely

$$r_n(x;\pm q^{\frac12},q^{\alpha+\frac12},-q^{\beta+\frac12}|q)=q^{-n\alpha}\frac{(q,-q^{\alpha+\beta+1};q)_n}{(q^{\alpha+1},-q^{\beta+1};q)_n}P_n^{(\alpha,\beta)}(x|q^2). \tag{11}$$

In fact, this Askey–Wilson polynomial is related by a quadratic transformation which follows from a formula due to Singh [2, (4.22)], [5, (III.21)], [11, (7)]

$${}_4\phi_3\left(\begin{matrix}q^{-2n},q^{2n}a^2,c^2,qb^2\\-a,-qa,q^2b^2c^2\end{matrix};q^2,q^2\right)=(bc)^n\frac{(-q,-\frac{a}{bc};q)_n}{(-a,-qbc;q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^na,\frac{c}{b},\frac{qb}{c}\\-q,-\frac{a}{bc},qbc\end{matrix};q,q\right), \tag{12}$$

or equivalently,

$${}_4\phi_3\left(\begin{matrix}q^{-n},q^na^2,c^2,q^{\frac12}b^2\\-a,-q^{\frac12}a,qb^2c^2\end{matrix};q,q\right)=(bc)^n\frac{(-q^{\frac12},-\frac{a}{bc};q^{\frac12})_n}{(-a,-q^{\frac12}bc;q^{\frac12})_n}\,{}_4\phi_3\left(\begin{matrix}q^{-\frac{n}{2}},q^{\frac{n}{2}}a,\frac{c}{b},\frac{q^{\frac12}b}{c}\\-q^{\frac12},-\frac{a}{bc},q^{\frac12}bc\end{matrix};q^{\frac12},q^{\frac12}\right), \tag{13}$$

but was also proved independently by Askey and Wilson in [2, (3.2)]), [9, (1.12)]

$$r_n\left(x;\pm q^{\frac12},q^{\alpha+\frac12},-q^{\beta+\frac12}|q\right)=q^{-n\alpha}\frac{(-q^{\alpha+1},-q^{\alpha+\beta+1};q)_n}{(-q^{\beta+1},-q;q)_n}r_n\left(x;q^{\alpha+\frac12\{\substack{1\\3}\}},-q^{\beta+\frac12\{\substack{1\\3}\}}|q^2\right). \tag{14}$$

The parity relation for continuous q-Jacobi polynomials is given by [8, (165)]

$$P_n^{(\alpha,\beta)}(-x|q)=-q^{\frac12n(\alpha-\beta)}P_n^{(\beta,\alpha)}(x|q). \tag{15}$$

The continuous q-Jacobi polynomials are orthogonal over $x=\frac12(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta})=\cos\theta\in(-1,1)$ which is demonstrated by the orthogonality relation

$$\int_{-1}^1 P_m^{(\alpha,\beta)}(x|q)P_n^{(\alpha,\beta)}(x|q)\,\frac{w(x;\alpha,\beta|q)}{\sqrt{1-x^2}}\,\mathrm{d}x=h_n(\alpha,\beta|q)\delta_{m,n}, \tag{16}$$

where the weight function and the norm for the continuous q-Jacobi polynomials are given by

$$w(x;\alpha,\beta|q)=\frac{(\mathrm{e}^{\pm2i\theta};q)_\infty}{(q^{\frac12\alpha+\frac14}\mathrm{e}^{\pm i\theta},q^{\frac12\alpha+\frac34}\mathrm{e}^{\pm i\theta},-q^{\frac12\beta+\frac14}\mathrm{e}^{\pm i\theta},-q^{\frac12\beta+\frac34}\mathrm{e}^{\pm i\theta};q)_\infty}, \tag{17}$$

$$h_n(\alpha,\beta|q)=\frac{2\pi q^{(\alpha+\frac12)n}(q^{\frac{\alpha+\beta+2}{2}},q^{\frac{\alpha+\beta+3}{2}};q)_\infty(q^{\alpha+1},q^{\beta+1},q^{\frac{\alpha+\beta+1}{2}};q)_n}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}};q)_\infty(q,q^{\alpha+\beta+1},q^{\frac{\alpha+\beta+3}{2}};q)_n}. \tag{18}$$

Theorem 1.2. *Let $q \in \mathbb{C}^\dagger$, $n \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{C}$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$. Then, the exhaustive list of all the balanced ${}_4\phi_3$ continuous q-Jacobi polynomial representations is*

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}z^{\pm} \\ q^{\alpha+1}, -q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}} \end{array};q,q\right) \tag{19}$$

$$= q^{-\frac{n}{2}}\frac{(q^{\alpha+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{\frac{1}{2}\alpha+\frac{3}{4}}z^{\pm} \\ q^{\alpha+1}, -q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{2\\3}\right\}} \end{array};q,q\right) \tag{20}$$

$$= \left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, -q^{\frac{1}{2}\beta+\frac{1}{4}}z^{\pm} \\ q^{\beta+1}, -q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}} \end{array};q,q\right) \tag{21}$$

$$= \left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, -q^{\frac{1}{2}\beta+\frac{3}{4}}z^{\pm} \\ q^{\beta+1}, -q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{2\\3}\right\}} \end{array};q,q\right); \tag{22}$$

$$P_n^{(\alpha,\beta)}(x|q)$$

$$= q^{-\binom{n}{2}}\left(-q^{-\frac{1}{2}}\right)^n\frac{\left(q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}}, q^{\frac{1}{2}\alpha+\frac{3}{4}}z^{\pm};q\right)_n}{(q,q^{\alpha+\beta+1};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{-\alpha-n}, -q^{-\frac{\alpha+\beta}{2}-n-\frac{1}{2}\left\{\substack{0\\1}\right\}} \\ q^{-\alpha-\beta-2n}, q^{-\frac{1}{2}\alpha+\frac{1}{4}-n}z^{\pm} \end{array};q,q\right) \tag{23}$$

$$= q^{-\binom{n}{2}}(-1)^n\frac{\left(q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}}, q^{\frac{1}{2}\alpha+\frac{1}{4}}z^{\pm};q\right)_n}{(q,q^{\alpha+\beta+1};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{-\alpha-n}, -q^{-\frac{\alpha+\beta}{2}-n+\frac{1}{2}\left\{\substack{0\\1}\right\}} \\ q^{-\alpha-\beta-2n}, q^{-\frac{1}{2}\alpha+\frac{3}{4}-n}z^{\pm} \end{array};q,q\right) \tag{24}$$

$$= \left(q^{\frac{\alpha-\beta-n}{2}}\right)^n\frac{\left(q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}}, -q^{\frac{1}{2}\beta+\frac{3}{4}}z^{\pm};q\right)_n}{(q,q^{\alpha+\beta+1};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{-\beta-n}, -q^{-\frac{\alpha+\beta}{2}-n-\frac{1}{2}\left\{\substack{0\\1}\right\}} \\ q^{-\alpha-\beta-2n}, -q^{-\frac{1}{2}\beta+\frac{1}{4}-n}z^{\pm} \end{array};q,q\right) \tag{25}$$

$$= q^{-\binom{n}{2}}\left(q^{\frac{\alpha-\beta}{2}}\right)^n\frac{\left(q^{\frac{\alpha+\beta}{2}+\frac{1}{2}\left\{\substack{1\\2}\right\}}, -q^{\frac{1}{2}\beta+\frac{1}{4}}z^{\pm};q\right)_n}{(q,q^{\alpha+\beta+1};q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{-\beta-n}, -q^{-\frac{\alpha+\beta}{2}-n+\frac{1}{2}\left\{\substack{0\\1}\right\}} \\ q^{-\alpha-\beta-2n}, -q^{-\frac{1}{2}\beta+\frac{3}{4}-n}z^{\pm} \end{array};q,q\right); \tag{26}$$

and

$$P_n^{(\alpha,\beta)}(x|q) = z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(q^{\alpha+1}, -q^{\frac12\beta+\frac14\{\frac13\}}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta}{2}+\frac12\{\frac12\}};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{-\beta-n}, q^{\frac12\alpha+\frac14\{\frac13\}}z \\ q^{\alpha+1}, -q^{-\frac12\beta-n+\frac14\{\frac13\}}z \end{matrix}; q, q\right) \tag{27}$$

$$= z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(q^{\beta+1}, q^{\frac12\alpha+\frac14\{\frac13\}}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta}{2}+\frac12\{\frac12\}};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{-\alpha-n}, -q^{\frac12\beta+\frac14\{\frac13\}}z \\ q^{\beta+1}, q^{-\frac12\alpha-n+\frac14\{\frac13\}}z \end{matrix}; q, q\right) \tag{28}$$

$$= z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(q^{\frac{\alpha}{2}+\frac34}z^{-1}, -q^{\frac{\beta}{2}+\frac34}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta+2}{2}};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, -q^{-\frac{\alpha+\beta+1}{2}-n}, q^{\frac12\alpha+\frac14}z, -q^{\frac12\beta+\frac14}z \\ -q^{\frac{\alpha+\beta+1}{2}}, q^{-\frac12\alpha+\frac14-n}z, -q^{-\frac12\beta+\frac14-n}z \end{matrix}; q, q\right) \tag{29}$$

$$= z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(q^{\frac{\alpha}{2}+\frac34}z^{-1}, -q^{\frac{\beta}{2}+\frac14}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta+1}{2}};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, -q^{-\frac{\alpha+\beta}{2}-n}, q^{\frac12\alpha+\frac14}z, -q^{\frac12\beta+\frac34}z \\ -q^{\frac{\alpha+\beta+2}{2}}, q^{-\frac12\alpha+\frac14-n}z, -q^{-\frac12\beta+\frac34-n}z \end{matrix}; q, q\right) \tag{30}$$

$$= z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(q^{\frac{\alpha}{2}+\frac14}z^{-1}, -q^{\frac{\beta}{2}+\frac34}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta+1}{2}};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, -q^{-\frac{\alpha+\beta}{2}-n}, q^{\frac12\alpha+\frac34}z, -q^{\frac12\beta+\frac14}z \\ -q^{\frac{\alpha+\beta+2}{2}}, q^{-\frac12\alpha+\frac34-n}z, -q^{-\frac12\beta+\frac14-n}z \end{matrix}; q, q\right) \tag{31}$$

$$= z^n q^{(\frac{\alpha}{2}+\frac14)n} \frac{\left(-q^{\frac{\alpha+\beta+3}{2}}, q^{\frac{\alpha}{2}+\frac14}z^{-1}, -q^{\frac{\beta}{2}+\frac14}z^{-1};q\right)_n}{(q, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}};q)_n} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, -q^{-\frac{\alpha+\beta-1}{2}-n}, q^{\frac{\alpha}{2}+\frac34}z, -q^{\frac{\beta}{2}+\frac34}z \\ -q^{\frac{\alpha+\beta+3}{2}}, q^{-\frac{\alpha}{2}+\frac34-n}z, -q^{-\frac{\beta}{2}+\frac34-n}z \end{matrix}; q, q\right), \tag{32}$$

and as well as the application of the map $z \mapsto z^{-1}$ in (27)–(32).

Proof. In [3, Theorem 7, (13)–(15)] a description of all balanced ${}_4\phi_3$ representations of the Askey–Wilson polynomials are given with parameters $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$. By using (9) which is the description of the continuous q-Jacobi polynomials in terms of the Askey–Wilson polynomials, one is able to obtain an exhaustive list of all balanced ${}_4\phi_3$ representations of the continuous q-Jacobi polynomials. In line with (9), we choose

$$\mathbf{a} := \left\{q^{\frac12\alpha+\frac14}, q^{\frac12\alpha+\frac34}, -q^{\frac12\beta+\frac14}, -q^{\frac12\beta+\frac34}\right\}. \tag{33}$$

By evaluating all permutations of (33) in [3, (13), (14), (15)], one may obtain (19)–(22), (25)–(26), (27)–(32) respectively. Since the continuous q-Jacobi polynomials are a function of $x = \frac12(z+z^{-1})$, and (19)–(26) are invariant under the interchange of $z \mapsto z^{-1}$, one may as well apply this replacement to (27)–(32) to obtain six alternative representations of the continuous q-Jacobi polynomials. This completes the proof. ∎

Remark 1.3. *Note that one may also obtain an exhaustive list of all continuous q-Jacobi polynomial terminating ${}_8W_7$ representations by applying the above procedure to [3, (16)–(19)]. However, due to space limitations, we will omit this computation for the present work. We might also add that by counting the members of the equivalence classes of terminating ${}_4\phi_3$ representations of the continuous q-Jacobi polynomials, one may explore the symmetry group of these transformations which should necessarily be a subgroup of the symmetric group S_6, the symmetry group of the terminating representations of the Askey–Wilson polynomials, see [3, Section 4], and references therein.*

2 Special values for the continuous q-Jacobi polynomials

For the following special values for the argument of the continuous q-Jacobi polynomials, we are able to re-express them in terms of Askey–Wilson polynomials with degree m as opposed to n.

Theorem 2.1. *Let $q \in \mathbb{C}^\dagger$, $n, m \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{C}$. The following continuous q-Jacobi polynomial specializations have alternative Askey–Wilson representations given as follows*

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac14+m}+q^{-\frac12\alpha-\frac14-m})|q)=\frac{(q^{\alpha+1};q)_n\left(q^{\frac12(\alpha+\beta+1)}\right)^m}{(q;q)_n(q^{\alpha+1},-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)};q)_m}$$
$$\times p_m(\tfrac12(q^{\frac{\alpha+\beta+1}{2}+n}+q^{-\frac{\alpha+\beta+1}{2}-n});q^{\frac12(\alpha+\beta+1)},q^{\frac12(\alpha-\beta+1)},-q^{\frac12},-1|q), \tag{34}$$

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac34+m}+q^{-\frac12\alpha-\frac34-m})|q)=\frac{q^{-\frac{n}{2}}\left(q^{\frac12(\alpha+\beta+1)}\right)^m(q^{\alpha+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n(q^{\alpha+1},-q^{\frac12(\alpha+\beta+2)},-q^{\frac12(\alpha+\beta+3)};q)_m}$$
$$\times p_m(\tfrac12(q^{\frac{\alpha+\beta+1}{2}+n}+q^{-\frac{\alpha+\beta+1}{2}-n});q^{\frac12(\alpha+\beta+1)},q^{\frac12(\alpha-\beta+1)},-q^{\frac12},-q|q), \tag{35}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac14+m}+q^{-\frac12\beta-\frac14-m})|q)=\frac{\left(-q^{\frac12(\alpha-\beta)}\right)^n\left(q^{\frac12(\alpha+\beta+1)}\right)^m(q^{\beta+1};q)_n}{(q;q)_n(q^{\beta+1},-q^{\frac12(\alpha+\beta+1)},-q^{\frac12(\alpha+\beta+2)};q)_m}$$
$$\times p_m(\tfrac12(q^{\frac{\alpha+\beta+1}{2}+n}+q^{-\frac{\alpha+\beta+1}{2}-n});q^{\frac12(\alpha+\beta+1)},q^{\frac12(\beta-\alpha+1)},-q^{\frac12},-1|q), \tag{36}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac34+m}+q^{-\frac12\beta-\frac34-m})|q)=\frac{\left(-q^{\frac12(\alpha-\beta-1)}\right)^n\left(q^{\frac12(\alpha+\beta+1)}\right)^m(q^{\beta+1},-q^{\frac12(\alpha+\beta+3)};q)_n}{(q,-q^{\frac12(\alpha+\beta+1)};q)_n(q^{\beta+1},-q^{\frac12(\alpha+\beta+2)},-q^{\frac12(\alpha+\beta+3)};q)_m}$$
$$\times p_m(\tfrac12(q^{\frac{\alpha+\beta+1}{2}+n}+q^{-\frac{\alpha+\beta+1}{2}-n});q^{\frac12(\alpha+\beta+1)},q^{\frac12(\beta-\alpha+1)},-q^{\frac12},-q|q). \tag{37}$$

Proof. Start by considering the Askey–Wilson polynomial representations of these continuous q-Jacobi polynomials (9) with these particular arguments. The specific arguments provided along with the identification that $x=\frac12(z+z^{-1})$ and therefore

$$z=z_m\in\{q^{\frac12\alpha+\frac14+m},q^{\frac12\alpha+\frac34+m},-q^{\frac12\beta+\frac14+m},-q^{\frac12\beta+\frac34+m}\}, \tag{38}$$

for (34)–(37) respectively. Providing the particular specializations of the argument produces the following ${}_4\phi_3$ representations

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{\alpha}{2}+\frac{1}{4}+m}+q^{-\frac{\alpha}{2}-\frac{1}{4}-m})|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{\alpha+\beta+1+n},q^{-m},q^{\alpha+\frac{1}{2}+m}\\ q^{\alpha+1},-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}}\end{matrix};q,q\right), \quad (39)$$

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{\alpha}{2}+\frac{3}{4}+m}+q^{-\frac{\alpha}{2}-\frac{3}{4}-m})|q)$$
$$=q^{-\frac{n}{2}}\frac{(q^{\alpha+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{\alpha+\beta+1+n},q^{-m},q^{\alpha+\frac{3}{2}+m}\\ q^{\alpha+1},-q^{\frac{\alpha+\beta+2}{2}},-q^{\frac{\alpha+\beta+3}{2}}\end{matrix};q,q\right), \quad (40)$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{\beta}{2}+\frac{1}{4}+m}+q^{-\frac{\beta}{2}-\frac{1}{4}-m})|q)$$
$$=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{\alpha+\beta+1+n},q^{-m},q^{m+\beta+\frac{1}{2}}\\ q^{\beta+1},-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}}\end{matrix};q,q\right), \quad (41)$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{\beta}{2}+\frac{3}{4}+m}+q^{-\frac{\beta}{2}-\frac{3}{4}-m})|q)$$
$$=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{\alpha+\beta+1+n},q^{-m},q^{\beta+\frac{3}{2}+m}\\ q^{\beta+1},-q^{\frac{\alpha+\beta+2}{2}},-q^{\frac{\alpha+\beta+3}{2}}\end{matrix};q,q\right). \quad (42)$$

It is then straightforward to convert these particular continuous q-Jacobi polynomials into a form where they can be represented as either Askey–Wilson polynomials in the degree n but also as Askey–Wilson polynomials in degree m. Solving for the particular values of the constants $\mathbf{a}$ in (3) for the degree m case completes the proof. ■

One, therefore, has the following special values for the continuous q-Jacobi polynomials which also follow directly from the identity (5).

Corollary 2.2. *Let $q\in\mathbb{C}^\dagger$, $n\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. Then*

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}})|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}, \quad (43)$$

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{3}{4}}+q^{-\frac{1}{2}\alpha-\frac{3}{4}})|q)=q^{-\frac{n}{2}}\frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(q,-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n}, \quad (44)$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{1}{4}}+q^{-\frac{1}{2}\beta-\frac{1}{4}})|q)=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n}, \quad (45)$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{3}{4}}+q^{-\frac{1}{2}\beta-\frac{3}{4}})|q)=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(q,-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n}. \quad (46)$$

Proof. Taking $m=0$ values in Theorem 2.1 completes the proof. ■

Remark 2.3. *One may inquire regarding the computation of perhaps product formulas for*

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}})|q), P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{3}{4}}+q^{-\frac{1}{2}\alpha-\frac{3}{4}})|q),$$
$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{1}{4}}+q^{-\frac{1}{2}\beta-\frac{1}{4}})|q), P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{3}{4}}+q^{-\frac{1}{2}\beta-\frac{3}{4}})|q),$$

using symmetry (15) and the special values for the Askey–Wilson polynomials given [8, (114)]. However, this is not possible since the interchange of $\alpha\leftrightarrow\beta$ with the invariance of the argument prevents the ${}_4\phi_3$s from being summable.

One may also express the results of Theorem 2.1 in terms of q-Racah polynomials (6).

Theorem 2.4. *Let $q\in\mathbb{C}^\dagger$, $n,m\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. Then the specialization for the continuous q-Jacobi polynomials there are the following special values given in terms of q-Racah polynomials as follows*

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac{\alpha}{2}+\frac14+m}+q^{-\frac{\alpha}{2}-\frac14 m})|q)$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}R_n(q^{\alpha+\frac12+m}+q^{-m};q^\alpha,q^\beta,-q^{\frac{\alpha+\beta}{2}},-q^{\frac{\alpha-\beta-1}{2}}|q) \tag{47}$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}R_m(q^{\alpha+\beta+1+n}+q^{-n};-q^{\frac{\alpha+\beta}{2}},-q^{\frac{\alpha-\beta-1}{2}},q^\alpha,q^\beta|q), \tag{48}$$

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac{\alpha}{2}+\frac34+m}+q^{-\frac{\alpha}{2}-\frac34-m})|q)$$

$$=q^{-\frac{n}{2}}\frac{(q^{\alpha+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_n(q^{\alpha+\frac32+m}+q^{-m};q^\alpha,q^\beta,-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha-\beta}{2}}|q) \tag{49}$$

$$=q^{-\frac{n}{2}}\frac{(q^{\alpha+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_m(q^{\alpha+\beta+1+n}+q^{-n};-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha-\beta}{2}},q^\alpha,q^\beta|q), \tag{50}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac{\beta}{2}+\frac14+m}+q^{-\frac{\beta}{2}-\frac14-m})|q)$$

$$=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n}R_n(q^{\beta+\frac12+m}+q^{-m};q^\beta,q^\alpha,-q^{\frac{\beta-\alpha-1}{2}},-q^{\frac{\alpha+\beta}{2}}|q) \tag{51}$$

$$=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n}R_m(q^{\alpha+\beta+1+n}+q^{-n};-q^{\frac{\alpha+\beta}{2}},-q^{\frac{\beta-\alpha-1}{2}},q^\beta,q^\alpha|q), \tag{52}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac{\beta}{2}+\frac34+m}+q^{-\frac{\beta}{2}-\frac34-m})|q)$$

$$=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_n(q^{\beta+\frac32+m}+q^{-m};q^\beta,q^\alpha,-q^{\frac{\beta+\alpha+1}{2}},-q^{\frac{\beta-\alpha}{2}}|q) \tag{53}$$

$$=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q,-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_m(q^{\alpha+\beta+1+n}+q^{-n};-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\beta-\beta}{2}},q^\beta,q^\alpha|q). \tag{54}$$

Proof. Start with Theorem 2.1 and utilize (39)–(42) with (6) to write these hypergeometric representations in terms of q-Racah polynomials. This completes the proof. ∎

Corollary 2.5. *Let $q\in\mathbb{C}^\dagger$, n,$m\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. If $\beta=-\frac12$, $\beta=\frac12$, $\alpha=-\frac12$, $\alpha=\frac12$, respectively, for the following specialized continuous q-Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac14+m}+q^{-\frac12\alpha-\frac14-m})|q), P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12\alpha+\frac34+m}+q^{-\frac12\alpha-\frac34-m})|q),$$
$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac14+m}+q^{-\frac12\beta-\frac14-m})|q), P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12\beta+\frac34+m}+q^{-\frac12\beta-\frac34-m})|q),$$

then one has the following duality relations

$$R_n(q^{\alpha+\frac12+m}+q^{-m};q^\alpha,q^{-\frac12},-q^{\frac12\alpha\pm\frac14}|q)=R_m(q^{\alpha+\frac12+n}+q^{-n};q^\alpha,q^{-\frac12},-q^{\frac12\alpha\pm\frac14}|q). \tag{55}$$

Proof. Replacing $\beta=\pm\frac12$, $\alpha=\pm\frac12$ in Theorem 2.4 completes the proof. ∎

Now consider the continuous q-Jacobi polynomials with special argument

$$x_m^{\pm} := \pm\tfrac{1}{2}(q^{\frac14+\frac{m}{2}}+q^{-\frac14-\frac{m}{2}}), \tag{56}$$

which are given in terms the Askey–Wilson polynomials through (9) as

$$P_n^{(\alpha,\beta)}(x_m^{\pm}|q) = \frac{\left(q^{\frac12\alpha+\frac14}\right)^n}{(q,-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}};q)_n} p_n(x_m^{\pm};q^{\frac{\alpha}{2}+\frac14\{\substack{1\\3}\}},-q^{\frac{\beta}{2}+\frac14\{\substack{1\\3}\}}|q). \tag{57}$$

The points $x_m^{\pm}$ are points where special values occur for the continuous q-Jacobi polynomials. Special values for special functions and orthogonal polynomials are rare, so it is important to recognize these, and then take advantage of them if one can. This is what we will do in Section 3 where we utilize these special values to compute new identities for basic hypergeometric series. Although we apply these special values to the Poisson kernel for these polynomials, it is quite possible to apply them to almost any identity for these polynomials. We consider some of the properties of the continuous q-Jacobi polynomials with this argument. First, we show some alternative Askey–Wilson representations of these polynomials with the special argument $x_m^{\pm}$.

Theorem 2.6. *Let $q\in\mathbb{C}^{\dagger}$, $m,n\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. Then*

$$P_n^{(\alpha,\beta)}(x_m^{\pm}|q) = \frac{\left(\pm q^{\frac12\alpha+\frac14}\right)^n}{(q^{\frac12},-q^{\frac12},-q^{\frac{\alpha+\beta+1}{2}};q^{\frac12})_n} p_n(x_m^{\pm};q^{\frac14},-q^{\frac14},\pm q^{\frac{\alpha}{2}+\frac14},\mp q^{\frac{\beta}{2}+\frac14}|q^{\frac12}), \tag{58}$$

$$= \frac{\left(\pm q^{\frac12\alpha}\right)^n\left(q^{\frac{\alpha+\beta+1}{4}}\right)^m(\pm q^{\frac{\alpha+1}{2}},\mp q^{\frac{\beta+1}{2}};q^{\frac12})_n}{(q^{\frac12},-q^{\frac{\alpha+\beta+1}{2}};q^{\frac12})_n(-q^{\frac12},\pm q^{\frac{\alpha+1}{2}},\mp q^{\frac{\beta+1}{2}};q^{\frac12})_m}$$
$$\times p_m(\tfrac12(q^{\frac{\alpha+\beta+1+2n}{4}}+q^{-\frac{\alpha+\beta+1+2n}{4}});q^{\frac{\alpha+\beta+1}{4}},-q^{\frac{-\alpha-\beta+1}{4}},\pm q^{\frac{\alpha-\beta+1}{4}},\mp q^{\frac{\beta-\alpha+1}{4}}|q^{\frac12}). \tag{59}$$

Proof. Start with the representation of the continuous q-Jacobi polynomials in terms of the Askey–Wilson polynomials (9) with special argument $x_m^{\pm}$ (56), namely (57). Now consider $x=\frac12(z+z^{-1})$, therefore $z=z_m:=\pm q^{\frac14+\frac{m}{2}}$ in (19) and replacing $q\mapsto q^2$ one obtains,

$$P_n^{(\alpha,\beta)}(\pm\tfrac12(q^{\frac12+m}+q^{-\frac12-m})|q^2) = \frac{(q^{2\alpha+2};q^2)_n}{(q^2;q^2)_n}$$
$$\times {}_4\phi_3\left(\begin{matrix}q^{-2n},q^{2\alpha+2\beta+2+2n},\pm q^{\alpha+1+m},\pm q^{\alpha-m}\\ q^{2\alpha+2},-q^{\alpha+\beta+1},-q^{\alpha+\beta+2}\end{matrix};q^2,q^2\right). \tag{60}$$

Applying the quadratic transformation (12) produces

$$P_n^{(\alpha,\beta)}(\pm\tfrac12(q^{\frac12+m}+q^{-\frac12-m})|q^2) = (\pm q^{\alpha})^n\frac{(\pm q^{\alpha+1},\mp q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n}{}_4\phi_3\left(\begin{matrix}q^{-n},q^{\alpha+\beta+1+n},q^{-m},q^{m+1}\\ \pm q^{\alpha+1},\mp q^{\beta+1},-q\end{matrix};q,q\right),$$

which can be viewed as either an Askey–Wilson polynomial with degree n or m. Obtaining these representations through (3), and then replacing $q^2\mapsto q$, completes the proof. ■

Remark 2.7. *Note that the above proof could be accomplished directly using (13) in (19). Furthermore one should observe that of all the ${}_4\phi_3$ representations (19)–(32), only (19) and (21) allow for the quadratic transformation (12) (or (13)). However, if one starts with (21) in order to prove Theorem 2.6, one arrives at an identical result. The fact that (19) and (21) are the only representations which satisfy the quadratic transformation (12) (or (13)) can be seen since (23)–(21) do not contain the necessary q^n numerator entry and (20), (22) do not satisfy the conditions given in (13).*

Now consider the $m=0,1$ special cases. This leads to the following result.

Corollary 2.8. *Let $q\in\mathbb{C}^\dagger$, $n\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. Then*

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12}+q^{-\frac12})|q^2)=(q^\alpha)^n\,\frac{(q^{\alpha+1},-q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},\tag{61}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12}+q^{-\frac12})|q^2)=(-q^\alpha)^n\,\frac{(-q^{\alpha+1},q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},\tag{62}$$

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\alpha+\frac12}+q^{-\alpha-\frac12}|q^2)=\frac{(q^{\alpha+1},-q^{\alpha+1};q)_n}{(q,-q;q)_n},\tag{63}$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\beta+\frac12}+q^{-\beta-\frac12})|q^2)=\left(-q^{\alpha-\beta}\right)^n\frac{(q^{\beta+1},-q^{\beta+1};q)_n}{(q,-q;q)_n}.\tag{64}$$

Proof. For (61), (62) use the sum [2, (4.23)] due to the quadratic transformation for Askey–Wilson polynomials (12). ■

Corollary 2.9. *Let $q\in\mathbb{C}^\dagger$, $n\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$. Then one also has the following special value*

$$\begin{aligned}P_n^{(\alpha,\beta)}(\pm\tfrac12(q^{\frac34}+q^{-\frac34})|q)&=\left(\pm q^{\frac{\alpha}{2}}\right)^n\frac{(\pm q^{\frac{\alpha+1}{2}},\mp q^{\frac{\beta+1}{2}};q^{\frac12})_n}{(q^{\frac12},-q^{\frac{\alpha+\beta+1}{2}};q^{\frac12})_n}\\&\qquad\times\left(1-\frac{(1-q^{\frac12})(1-q^{-\frac{n}{2}})(1-q^{\frac{\alpha+\beta+n+1}{2}})}{(1\mp q^{\frac{\alpha+1}{2}})(1\pm q^{\frac{\beta+1}{2}})}\right).\end{aligned}\tag{65}$$

Proof. The proof follows in exactly the same way as for the argument $\pm\frac12(q^{\frac14}+q^{-\frac14})$, except instead of using the sum [2, (4.23)], directly use the quadratic transformation for Askey–Wilson polynomials [2, (4.22)]. Then with this transformation, the resulting terminating ${}_4\phi_3$ has a q^{-1} as one of the numerator parameters, so the sum truncates to the first two terms and the above specializations follow. As well, simply using Theorem 2.6 with $m=1$ in (59) completes the proof. ■

Theorem 2.10. *Let $q\in\mathbb{C}^\dagger$, $m,n,N\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$, and in the following special values for the continuous q-Jacobi polynomials for the positive sign in the argument, choose $\alpha=-N-1$ and leave β unrestricted and for the negative sign in the argument chose $\beta=-N-1$ and α unrestricted. Furthermore, for (66) let $n\le N$, and for (67) let $m\le N$. Then*

$$\begin{aligned}&P_n^{(\alpha,\beta)}(\pm\tfrac12(q^{\frac14+\frac{m}{2}}+q^{-\frac14-\frac{m}{2}})|q)\\&\quad=(\pm q^{\frac12\alpha})^n\frac{(\pm q^{\frac{\alpha+1}{2}},\mp q^{\frac{\beta+1}{2}};q)_n}{(q^{\frac12},-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_n(q^{-\frac{m}{2}}+q^{\frac{m+1}{2}};\pm q^{\frac12\alpha},\pm q^{\frac12\beta},-1,-1|q^{\frac12})\end{aligned}\tag{66}$$

$$\quad=(\pm q^{\frac12\alpha})^n\frac{(\pm q^{\frac{\alpha+1}{2}},\mp q^{\frac{\beta+1}{2}};q)_n}{(q^{\frac12},-q^{\frac{\alpha+\beta+1}{2}};q)_n}R_m(q^{-\frac{n}{2}}+q^{\frac{\alpha+\beta+n+1}{2}};-1,-1,\pm q^{\frac12\alpha},\pm q^{\frac12\beta}|q).\tag{67}$$

Proof. Start with expression (60), and the definition of the q-Racah polynomials (6). We choose $\{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta\}=\{\pm q^\alpha,\pm q^\beta,-1,-1\}$, and therefore $\mu(m)=q^{m+1}+q^{-m}$. Since $\bar\delta=\bar\gamma=-1$, the third condition in (7) is impossible, so we must use either one of the first two conditions. The only solution is for the positive argument to choose $\alpha=-N-1$ and leave β unrestricted and for the negative argument to chose $\beta=-N-1$ and to leave α unrestricted. Next, consider (67). We choose $\{\bar\alpha,\bar\beta,\bar\gamma,\bar\delta\}=\{-1,-1,\pm q^\alpha,\pm q^\beta\}$, and therefore $\mu(n)=q^{\alpha+\beta+n+1}+q^{-n}$.

Since $\bar{\alpha} = \bar{\beta} = -1$, the first condition in (7) is impossible, so we must use either one of the second or third conditions. Again, the only solution is for the positive argument to choose $\alpha = -N-1$ and leave β unrestricted and for the negative argument to chose $\beta = -N-1$ and to leave α unrestricted. Since these results are for $P_n^{(\alpha,\beta)}(x|q^2)$, mapping $q^2 \mapsto q$, completes the proof. ■

Setting $\beta = -\alpha$ in Theorem 2.10 produces the following result.

Corollary 2.11. *Let $q \in \mathbb{C}^\dagger$, $m, n \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{C}$. Then one has the following duality relations for the specialized continuous q-Jacobi polynomials*

$$P_n^{(\alpha,-\alpha)}(\pm\tfrac{1}{2}(q^{\frac{1}{4}+\frac{m}{2}} + q^{-\frac{1}{4}-\frac{m}{2}})|q), \tag{68}$$

namely

$$R_n(q^{-\frac{m}{2}} + q^{\frac{m+1}{2}}; \pm q^{\frac{1}{2}\alpha}, \pm q^{-\frac{1}{2}\alpha}, -1, -1|q) = R_m(q^{-\frac{n}{2}} + q^{\frac{n+1}{2}}; \pm q^{\frac{1}{2}\alpha}, \pm q^{-\frac{1}{2}\alpha}, -1, -1|q). \tag{69}$$

Proof. Replace $\alpha + \beta = 0$ in Theorem 2.10 completes the proof. ■

3 Special values for the continuous q-Jacobi polynomial Poisson kernel

In general, one may substitute special values of an orthogonal polynomial into an identity for these polynomials to obtain new specialized identities. We will now focus on one such application. We now utilize the special values which we derived in Section 1.3 within the context of the Poisson kernel for continuous q-Jacobi polynomials. This utilization of special values for basic hypergeometric orthogonal polynomials is both an application to the theory of basic hypergeometric orthogonal polynomials and nonterminating basic hypergeometric functions. The Poisson kernel for an orthogonal polynomial sequence is a bilinear generating function and is a function of the two variables, $x = \frac{1}{2}(z + z^{-1})$ and $y = \frac{1}{2}(w + w^{-1})$ (as well as a power series parameter $|t| < 1$ and other parameters involved). Inserting special values z, w (which correspond to x, y) in a Poisson kernel results in the conversion of the bilinear generating function. As we will see below, replacing either z or w in the Poisson kernel converts it to a generating function. Furthermore, replacing both z and w in the Poisson kernel converts it to a transformation formula for an arbitrary argument (perhaps subject to certain constraints) nonterminating basic hypergeometric function.

The most general Poisson kernel for Askey–Wilson polynomials $\mathsf{K}_t(x,y) := \mathsf{K}_t(x,y;\mathbf{a}|q)$, which is given by

$$\mathsf{K}_t(x,y) = \sum_{n=0}^{\infty} \frac{(\frac{abcd}{q}, \pm\sqrt{qabcd}; q)_n \, p_n(x;\mathbf{a}|q) p_n(y;\mathbf{a}|q) \, t^n}{(q, \pm\sqrt{\frac{abcd}{q}}, ab, ac, ad, bc, bd, cd; q)_n} \tag{70}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{abcd}{q}, \pm\sqrt{qabcd}, ab, ac, ad; q)_n t^n}{(q, \pm\sqrt{\frac{abcd}{q}}, bc, bd, cd; q)_n a^{2n}} r_n(x;\mathbf{a}|q) r_n(y;\mathbf{a}|q), \tag{71}$$

where we (and as well Gasper & Rahman (1986) [4]) have also used the normalized version of the Askey–Wilson polynomials (4). For the special case of the Askey–Wilson polynomials where

$ad=bc$, then the Poisson kernel takes a more simplified form, $K_t(x,y):=\mathsf{K}_t(x,y;\{a,b,c,\frac{bc}{a}\}|q)$ namely

$$K_t(x,y):=\sum_{n=0}^{\infty}\frac{(\frac{b^2c^2}{q},\pm q^{\frac12}bc,ab,ac;q)_nt^n}{(q,\pm q^{-\frac12}bc,\frac{b^2c}{a},\frac{bc^2}{a};q)_na^{2n}}r_n(x;\mathbf{a}|q)r_n(y;\mathbf{a}|q). \tag{72}$$

Gasper & Rahman [4, (6.13)] proved a very useful form of this Poisson kernel for Askey–Wilson polynomials with parameters (a,b,c,d) with the extra constraint $ad=bc$. This Poisson kernel is given in three terms, each term given as an infinite sum over a very-well poised and balanced ${}_{10}W_9$. One can see that for the continuous q-Jacobi polynomials (9) with the particular choice of $\{a,b,c,d\}$ in (10), the condition $ad=bc$ is satisfied. Then the Poisson kernel for continuous q-Jacobi polynomials is given by [4, (2.10)]

$$K_t(x,y):=K_t^{(\alpha,\beta)}(x,y|q)=\sum_{n=0}^{\infty}\frac{(q,q^{\alpha+\beta+1},q^{\frac{\alpha+\beta+3}{2}};q)_nt^n}{(q^{\alpha+1},q^{\beta+1},q^{\frac{\alpha+\beta+1}{2}};q)_nq^{(\alpha+\frac12)n}}P_n^{(\alpha,\beta)}(x|q)P_n^{(\alpha,\beta)}(y|q). \tag{73}$$

Replacing $\{a,b,c,d\}$ as in (10) produces the following form for the Poisson kernel for continuous q-Jacobi polynomials.

Theorem 3.1. *Let $q\in\mathbb{C}^\dagger$, $t,\alpha,\beta\in\mathbb{C}$, $|t|<1$, $x=\frac12(z+z^{-1})\in\mathbb{C}$, $y=\frac12(w+w^{-1})\in\mathbb{C}$. Then the symmetric Poisson kernel for continuous q-Jacobi polynomials is given by*

$$\begin{aligned}
K_t(x,y)=(1-t^2)&\frac{(-q^{\frac{\alpha+\beta+4}{2}}t;q)_\infty}{(-q^{\frac{-\alpha-\beta-2}{2}}t;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{\frac{\alpha+\beta+2}{2}},\pm q^{\frac{\alpha+\beta+3}{2}},-q^{\frac12\beta+\frac34}z^{\pm},-q^{\frac12\beta+\frac34}w^{\pm};q)_nq^n}{(q,q^{\beta+1},-q^{\frac12(\alpha+\beta+\{{2\atop 3}\})},-q^{\frac{\beta-\alpha+1}{2}},-q^{\frac{\alpha+\beta}{2}+2}t^{\pm};q)_n}\\
&\times{}_{10}W_9\Big(-q^{\frac{\alpha-\beta-2n-1}{2}};q^{-n},-q^{\frac{-\alpha-\beta-2n-1}{2}},q^{-\beta-n},q^{\frac12\alpha+\frac14}z^{\pm},q^{\frac12\alpha+\frac14}w^{\pm};q,q\Big)\\
&+\frac{(q^{\alpha+\beta+2},t,-q^{\frac{\alpha-\beta}{2}}t,q^{\frac12\alpha+\frac34}z^{\pm},q^{\frac12\alpha+\frac14}w^{\pm},-q^{\frac12\beta+\frac14}tz^{\pm},-q^{\frac12\beta+\frac34}tw^{\pm};q)_\infty}{(q^{\alpha+1},-q^{\frac{\alpha+\beta}{2}+\left\{\begin{smallmatrix}1/2\\1\\3/2\end{smallmatrix}\right\}},-q^{\frac{\alpha-\beta}{2}},q^{\beta+1}t,-q^{\frac{\alpha+\beta+2}{2}}t^{-1},tz^{\pm}w^{\pm};q)_\infty}\\
&\times\sum_{n=0}^{\infty}\frac{(-t,\pm\sqrt{q}t,q^{\beta+1}t,tz^{\pm}w^{\pm};q)_nq^n}{(q,-q^{\frac{-\alpha-\beta}{2}}t,qt^2,-q^{\frac{\alpha-\beta}{2}}t,-q^{\frac12\beta+\frac14}tz^{\pm},-q^{\frac12\beta+\frac34}tw^{\pm};q)_n}\\
&\times{}_{10}W_9\Big(q^{\beta+n}t;q^nt,-q^{\frac{\alpha+\beta+2n}{2}}t,-q^{\frac{\beta-\alpha+2n}{2}}t,-q^{\frac12\beta+\frac34}z^{\pm},-q^{\frac12\beta+\frac14}w^{\pm};q,q\Big)\\
&+\frac{(q^{\alpha+\beta+2},t,-q^{\frac{\beta-\alpha}{2}}t,-q^{\frac12\beta+\frac34}z^{\pm},-q^{\frac12\beta+\frac14}w^{\pm},q^{\frac12\alpha+\frac14}tz^{\pm},q^{\frac12\alpha+\frac34}tw^{\pm};q)_\infty}{(q^{\beta+1},-q^{\frac{\alpha+\beta}{2}+\left\{\begin{smallmatrix}1/2\\1\\3/2\end{smallmatrix}\right\}},-q^{\frac{\beta-\alpha}{2}},q^{\alpha+1}t,-q^{\frac12(\alpha+\beta+2)}t^{-1},tz^{\pm}w^{\pm};q)_\infty}\\
&\times\sum_{n=0}^{\infty}\frac{(-t,\pm\sqrt{q}t,q^{\alpha+1}t,tz^{\pm}w^{\pm};q)_nq^n}{(q,-q^{\frac{-\alpha-\beta}{2}}t,-q^{\frac{\beta-\alpha}{2}}t,qt^2,q^{\frac12\alpha+\frac14}tz^{\pm},q^{\frac12\alpha+\frac34}tw^{\pm};q)_n}\\
&\times{}_{10}W_9\Big(q^{\alpha+n}t;q^nt,-q^{\frac{\alpha+\beta+2n}{2}},-q^{\frac{\alpha-\beta+2n}{2}}t,q^{\frac12\alpha+\frac34}z^{\pm},q^{\frac12\alpha+\frac14}w^{\pm};q,q\Big).
\end{aligned} \tag{74}$$

Proof. This is obtained by making the replacement $\{a,b,c,d\}\mapsto\Big\{q^{\frac12\alpha+\frac14\{{1\atop 3}\}},-q^{\frac12\beta+\frac14\{{1\atop 3}\}}\Big\}$, namely (10), into Gasper & Rahman [4, (6.13)]. ■

Note that there is an evident symmetry under the replacement $z\mapsto z^{-1}$ and $w\mapsto w^{-1}$ in (74). Therefore there are 8 possibilities for substitutions, namely $z,w\in\{a,b,c,d\}$. One simple application of the special values given in Corollary 2.2, is that if you apply these substitutions to one of the continuous q-Jacobi polynomials in the bilinear generating function given by its Poisson kernel, it is converted to a generating function. Another easy application of these special values is that if you apply them to both of the continuous q-Jacobi polynomials, then the Poisson kernel is converted to a transformation formula for an arbitrary argument basic hypergeometric series. These kind of transformations are fairly rare in the literature of basic hypergeometric functions. The appearance of the arbitrary argument in the transformation formula comes from the power series parameter t, which appears in the bilinear generating function.

3.1 Generating functions that condense from $K_t(x, y)$

There is no evident symmetry in (74) with respect to a, b, c, d, so there are many choices of replacing with z or w using the special values a, b, c, d given in Corollary 2.2. Upon experimentation, we found that for two particular choices one obtains some interesting generating functions for continuous q-Jacobi polynomials, which also have interesting $q \to 1^-$ limits.

3.1.1 The generating function when $w = a$

Given $y = \frac{1}{2}(w + w^{-1})$, for the choice $w = a$ we obtain the following interesting generating function for continuous q-Jacobi polynomials.

Theorem 3.2. *Let $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $\alpha, \beta, t \in \mathbb{C}^*$ such that $|q^{\alpha+\frac{1}{2}}t| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta+1}, q^{\frac{1}{2}(\alpha+\beta+3)}; q)_n}{(q^{\beta+1}, q^{\frac{1}{2}(\alpha+\beta+1)}; q)_n} t^n P_n^{(\alpha,\beta)}(x|q)$$
$$= \frac{(-q^{\frac{1}{2}(3\alpha+\beta+5)}t, q^{2\alpha+1}t^2, -q^{\frac{1}{2}(\alpha+\beta+2)}t, q^{\alpha+\frac{1}{2}\beta+\frac{7}{4}}tz^{\pm}; q)_\infty}{(q^{2\alpha+2}t^2, -q^{\alpha+\beta+\frac{5}{2}}t, -q^{\alpha+1}t, q^{\frac{1}{2}\alpha+\frac{1}{4}}tz^{\pm}; q)_\infty}$$
$$\times {}_8W_7\left(-q^{\alpha+\beta+\frac{3}{2}}t; -q^{\alpha+\frac{3}{2}}t, q^{\frac{1}{2}(\beta-\alpha)}, q^{\frac{1}{2}(\alpha+\beta+3)}, -q^{\frac{1}{2}\beta+\frac{3}{4}}z^{\pm}; q, -q^{\alpha+\frac{1}{2}}t\right). \tag{75}$$

Proof. Start with (74) and replacing $w = a$ in Corollary 2.2 and the result follows. ■

This generating function has a $q \to 1^-$ limit given as follows.

Corollary 3.3. *Let $x, \alpha, \beta \in \mathbb{C}$, $|t| < 1$. Then one has the following generating functions for Jacobi polynomials, namely*

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n(\frac{1}{2}(\alpha+\beta+3))_n}{(\beta+1)_n(\frac{1}{2}(\alpha+\beta+1))_n} t^n P_n^{(\alpha,\beta)}(x)$$
$$= \frac{1-t^2}{(1+t^2-2tx)^{\frac{1}{2}(\alpha+\beta+3)}} {}_2F_1\left(\begin{matrix}\frac{1}{2}(\beta-\alpha), \frac{1}{2}(\alpha+\beta+3)\\ \beta+1\end{matrix}; \frac{-2t(x+1)}{1+t^2-2tx}\right)$$
$$= \frac{2^{\frac{1}{2}\beta}(1-t)\Gamma(\beta+1)}{(t(1+x))^{\frac{1}{2}\beta}(1+t^2-2tx)^{\frac{1}{2}\alpha+1}} P_{\alpha+1}^{-\beta}\left(\frac{1+t}{\sqrt{1+t^2-2tx}}\right), \tag{76}$$

where P_ν^μ is an associated Legendre function of the first kind [10, (14.3.6)].

Proof. This generating function is obtained by setting $w = a = q^{\frac{1}{2}\alpha+\frac{1}{4}}$ in (75) and then simplifying the resulting expression. ■

As a special case, this generating function leads to the following generating function for continuous q-ultraspherical/Rogers polynomials.

Corollary 3.4. *Let $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $\beta, t \in \mathbb{C}^*$ such that $|t| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{(q\beta;q)_n}{(\beta;q)_n} t^n C_n(x;\beta|q) = (1-\beta t^2) \frac{(q\beta t z^{\pm};q)_\infty}{(tz^{\pm};q)_\infty} \tag{77}$$

Proof. Starting with (75) with $\alpha = \beta$ causes the ${}_8W_7$ to become unity. Then using [7, p. 473]

$$P_n^{(\alpha,\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q^{2\alpha+1};q)_n} q^{(\frac{\alpha}{2}+\frac{1}{4})n} C_n(x; q^{\alpha+\frac{1}{2}}|q), \tag{78}$$

completes the proof. ■

Remark 3.5. *The $q \to 1^-$ Gegenbauer polynomial limit of the generating function given by Corollary 3.4 is*

$$\sum_{n=0}^{\infty} \frac{(1+\beta)_n}{(\beta)_n} t^n C_n^\beta(x) = \sum_{n=0}^{\infty} \frac{\beta+n}{\beta} t^n C_n^\beta(x) = \frac{1-t^2}{(1+t^2-2tx)^{\beta+1}}. \tag{79}$$

3.1.2 The generating function when $z = d$

Given $x = \frac{1}{2}(z + z^{-1})$, for the choice $z = d$ we obtain the following interesting generating function for continuous q-Jacobi polynomials.

Theorem 3.6. *Let $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $\alpha, \beta, t \in \mathbb{C}^*$ such that $|q^{\alpha+\frac{1}{2}}t| < 1$. Then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta+1}, \pm q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(q^{\alpha+1}, \pm q^{\frac{1}{2}(\alpha+\beta+1)};q)_n} t^n P_n^{(\alpha,\beta)}(x|q) \\
&\quad = \frac{(q^{\alpha+\beta+3}t, q^{\alpha+\beta+2}t^2;q)_\infty}{(t, q^{\alpha+\beta+3}t^2;q)_\infty} \, {}_5\phi_4\left(\begin{matrix} q^{\frac{1}{2}(\alpha+\beta+2)}, \pm q^{\frac{1}{2}(\alpha+\beta+3)}, q^{\frac{1}{2}\alpha+\frac{1}{4}}z^{\pm} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, q^{\alpha+\beta+3}t, \frac{q}{t} \end{matrix}; q, q\right) \\
&\qquad + \frac{(q^{\alpha+\beta+2}, -q^{\frac{1}{2}(\alpha+\beta+1)}t, -q^{\frac{1}{2}(\alpha+\beta+2)}t, q^{\alpha+1}t, q^{\frac{1}{2}\alpha+\frac{1}{4}}z^{\pm};q)_\infty}{(q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}, \frac{1}{t}, q^{\frac{1}{2}\alpha+\frac{1}{4}}tz^{\pm};q)_\infty} \\
&\qquad\qquad \times {}_5\phi_4\left(\begin{matrix} q^{\frac{1}{2}(\alpha+\beta+2)}t, \pm q^{\frac{1}{2}(\alpha+\beta+3)}t, q^{\frac{1}{2}\alpha+\frac{1}{4}}tz^{\pm} \\ -q^{\frac{1}{2}(\alpha+\beta+1)}t, qt, q^{\alpha+1}t, q^{\alpha+\beta+3}t^2 \end{matrix}; q, q\right).
\end{aligned} \tag{80}$$

Proof. This generating function is obtained by setting $z = d = -q^{\frac{1}{2}\beta+\frac{3}{4}}$ in (75), replacing $y = \frac{1}{2}(w + w^{-1})$ with x and then simplifying the resulting expression completes the proof. ■

This generating function for continuous q-Jacobi polynomials has a $q \to 1^-$ limit given as follows.

Corollary 3.7. *Let $x, \alpha, \beta \in \mathbb{C}$, $|t| < 1$. Then, one has the following generating function for*

Jacobi polynomials, namely

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n(\frac12(\alpha+\beta+3))_n}{(\alpha+1)_n(\frac12(\alpha+\beta+1))_n}t^nP_n^{(\alpha,\beta)}(x)$$
$$=\frac{1-t^2}{(1-t)^{\alpha+\beta+3}}\,{}_2F_1\left(\begin{matrix}\frac12(\alpha+\beta+2),\frac12(\alpha+\beta+3)\\ \alpha+1\end{matrix};\frac{2t(y-1)}{(1-t)^2}\right)$$
$$=\frac{2^{\frac12\alpha}(1+t)\Gamma(\alpha+1)}{(t(y-1))^{\frac12\alpha}(1+t^2-2ty)^{\frac12\beta+1}}P_{\beta+1}^{-\alpha}\left(\frac{1-t}{\sqrt{1+t^2-2ty}}\right),\tag{81}$$

where P_ν^μ is an associated Legendre function of the first kind [10, (14.3.6)].

Starting from (80), using the symmetric limit $\alpha=\beta$, we can produce a new generating function for continuous q-ultraspherical polynomials.

Corollary 3.8. *Let $q\in\mathbb{C}^\dagger$, $x=\frac12(z+z^{-1})\in\mathbb{C}$, $\beta,t\in\mathbb{C}^*$ such that $|t|<1$. Then*

$$\sum_{n=0}^{\infty}\frac{(\pm q\beta;q)_n}{(\pm\beta;q)_n}t^nC_n(x;\beta|q)=\frac{(q\beta,-q\beta tz^{\pm},\beta t^2;q)_\infty}{(-\beta,tz^{\pm},-q\beta t^2;q)_\infty}\,{}_8W_7\left(-\beta t^2;-\frac1q,\pm q\sqrt{\beta}t,tz^{\pm};q,q\beta\right).\tag{82}$$

Proof. Starting with (80) with $\alpha=\beta$ causes the sum of two ${}_5\phi_4(q,q)$s to become a sum of two ${}_4\phi_3(q,q)$s which then naturally transforms to an ${}_8W_7$ using Bailey's transformation [10, (17.9.16)]. Then using (78) completes the proof. ∎

3.2 An arbitrary argument transformation that condenses from $K_t(x,y)$

As mentioned just below (74), if one makes the double replacement of z and w in the Poisson kernel, one obtains an arbitrary argument transformation formula. Because of limitations on space in the current manuscript, we will only treat a single example. Note however, that if one considers the Poisson kernel for continuous q-Jacobi polynomials, and then one takes the special values (43)–(46) for z and w simultaneously, then there are $4\times4=16$ combinations. For each case, the Poisson kernel reduces to a single nonterminating basic hypergeometric series with an arbitrary argument. The cases $z=w$ correspond to 4 unique transformations, and the 12 off-diagonal combinations combine into pairs, with 6 paired transformations for a single nonterminating basic hypergeometric series. Below, we present the results for the computation of these transformations. For the $z=w=a$ case, then the Poisson kernel produces a well-poised ${}_3\phi_2$ with arbitrary argument z. This is a well-poised nonterminating ${}_3\phi_2$ with arbitrary argument z expressed as a sum of two ${}_4\phi_3$s with argument q or a nonterminating ${}_8W_7$.

Theorem 3.9. *Let $q\in\mathbb{C}^\dagger$, $z,a,b\in\mathbb{C}$ such that $|a|,|z|<1$. Then the following well-poised ${}_3\phi_2$ is given by*

$${}_3\phi_2\left(\begin{matrix}ab,\sqrt{q}a,q\sqrt{ab}\\ \sqrt{q}b,\sqrt{ab}\end{matrix};q,z\right)=\frac{(a^2z^2,q\sqrt{ab}z,-\sqrt{qab}z,q\sqrt{a^3b}z,-\sqrt{q^3a^3b}z;q)_\infty}{(qa^2z^2,z,az,-\sqrt{q}az,-\sqrt{q^3}abz;q)_\infty}$$
$$\times\,{}_8W_7\left(-\sqrt{q}abz;q\sqrt{ab},-\sqrt{qab},\sqrt{\frac{b}{a}},-\sqrt{\frac{qb}{a}},-qaz;q,-az\right).\tag{83}$$

Proof. Start with the Poisson kernel for continuous q-Jacobi polynomials and substitute $z = w = a$. This converts the infinite series into a single nonterminating ${}_3\phi_2$. If one takes $q^{\alpha+\frac{1}{2}} \mapsto a$, $q^{\beta+\frac{1}{2}} \mapsto b$, then one arrives at a nonterminating transformation which is a sum of two ${}_4\phi_3$s with argument q. Using Bailey's transformation [10, (17.9.16)] we convert it to a nonterminating ${}_8W_7$. This completes the proof. ■

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