Extensions of discrete classical orthogonal polynomials beyond the orthogonality

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10th OPSFA. Leuven, Belgium

July 20th, 2009

Structure of the talk

The continuous case

- K. H. Kwon and L. L. Littlejohn, $(L_n^{(-k)})$ orthogonal w.r.t.

$$\langle f, g\rangle = (f(0), f'(0), \dots, f^{(k-1)}(0))A(g(0), g'(0), \dots, g^{(k-1)}(0))^T + \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx.$$

- The Jacobi polynomials $(P_n^{(-1,-1)})$ orthogonal w.r.t.

$$(f, g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x)dx.$$

- These examples suggest that COP with non-classical parameters can be provided with a orthogonality of Sobolev-type.
- Furthermore F. Marcellan and J.J. Moreno-Balcázar pointed out that a Sobolev-Askey tableau should be established.

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Basic properties

- Let (P_n) be a polynomial sequence and $\mathbf{u}$ be a linear functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\Delta(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1, \ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

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$$h_n^{\alpha,\beta}(x;N) = \frac{(-N, \alpha+1)_n}{(\alpha+\beta+n+1)_n} {}_3F_2\left(\begin{array}{c} -n, \alpha+\beta+n+1, -x \\ -N, \alpha+1 \end{array} \middle| 1 \right).$$

- Property of orthogonality.

$$\langle \mathbf{u}^H, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\Delta(x(\beta+N+1-x)\mathbf{u}^H) = ((\alpha+1)N - (\alpha+\beta+2)x)\mathbf{u}^H.$$

- Integral representation with some boundary condition:

$$\langle \mathbf{u}^H, P \rangle = \sum_{x=0}^{N} P(x) \frac{\Gamma(\beta+N+1-x)\Gamma(\alpha+1+x)}{\Gamma(1+x)\Gamma(N+1-x)}.$$

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$$h_{-1}^{\alpha,\beta}(x;N) = 0, \quad h_{0}^{\alpha,\beta}(x;N) = 1,$$
$$x h_{n}^{\alpha,\beta}(x;N) = h_{n+1}^{\alpha,\beta}(x;N) + \beta_n h_{n}^{\alpha,\beta}(x;N) + \gamma_n h_{n-1}^{\alpha,\beta}(x;N), \ \ n = 0,1,2,\ldots$$

where

$$\beta_n = \frac{(\alpha+1)N(\alpha+\beta) + n(2N-\alpha+\beta)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)},$$

$$\gamma_n = \frac{n(N+1-n)(\alpha+\beta+n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

Continuous Hahn polynomials

- Hypergeometric series:

$$p_n(x;a,b,c,d) = D_n\, {}_3F_2\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{array} \middle| 1 \right).$$

- Property of orthogonality: $\langle \mathbf{u}^{cH}, p_n p_m \rangle = d_n^2 \delta_{n,m}$.
- Distributional equation: $\delta f(x) = f(x+i/2) - f(x-i/2)$

$$\frac{\delta}{\delta x}((c-ix)(d-ix)\mathbf{u}^{cH}) = p_1(x;a,b,c,d)\mathbf{u}^{cH}.$$

- Integral representation with some boundary condition:

$$\langle \mathbf{u}^{cH}, P \rangle = \int_C P(z)\Gamma(a+iz)\Gamma(b+iz)\Gamma(c-iz)\Gamma(d-iz)\,dz,$$

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The limit relation *H-cH*

- The hypergeometric serie, re-written:

$$h_n^{\alpha,\beta}(x;N) = r_n \sum_{k=0}^{n} \frac{(-n, \alpha+\beta+n+1, -x)_k (-N+k)_{n-k}}{(\alpha+1,1)_k},$$

- The limit relation:

$$h_n^{\alpha,\beta}(x;N) = \lim_{\varepsilon \to 0} (-i)^n \; p_n(ix; 0, \beta+N+\varepsilon+1, -N-\varepsilon, \alpha+1).$$

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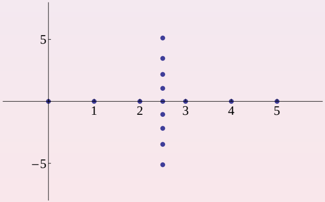
About the zeros of the extended Hahn polynomials



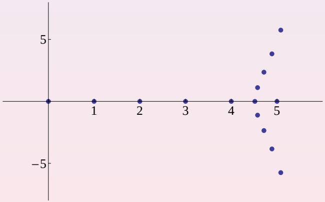


Figure: Zeros of $h_{15}^{1,1}(x;5)$ (left) and $h_{15}^{1,15}(x;5)$ (right)

The factorization

- For any integer k, $0 \leq k \leq n$,

$$\Delta^k h_n^{\alpha,\beta}(x;N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x;N-k),$$

$$ii.2) \qquad \nabla^k h_n^{\alpha,\beta}(x;N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x-k;N-k).$$

- The factorization:

$$h_n^{\alpha,\beta}(x;N) = (x-N)_{N+1}(-i)^{n-N-1} p_{n-N-1}(ix; N+1, \beta+N+1, 1, \alpha+1) = (x-N)_{N+1}(-i)^{n-N-1} p_{n-N-1}\left(\left(x-\frac{N}{2}\right)i; 1+\frac{N}{2}, \beta+1+ +\frac{N}{2}, 1+\frac{N}{2}, \alpha+1+\frac{N}{2}\right).$$

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$$\left.+\tfrac{N}{2},1+\tfrac{N}{2},\alpha+1+\tfrac{N}{2}\right).$$

A characterization Theorem for the Hahn polynomials

Theorem: Let N be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that: $-\alpha, -\beta \notin \{1, 2, \dots, N, N+2, \dots\}$, and $-\alpha - \beta \notin \{1, 2, \dots, 2N+1, 2N+3, \dots\}$. Then the family of Hahn polynomials is a OPS with respect to the Δ-Sobolev inner product:

$$(f, g)_S = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha,\beta}(x; N) + \int_C (\Delta^{N+1} f(z))(\Delta^{N+1} g(z))\omega^{\alpha,\beta}(z; N)dz,$$

where

$$\begin{aligned} \rho^{\alpha,\beta}(x; N) &= \frac{\Gamma(\beta + N + 1 - x)\Gamma(\alpha + x + 1)}{\Gamma(N + 1 - x)\Gamma(x + 1)}, \\ \omega^{\alpha,\beta}(z; N) &= \Gamma(-z)\Gamma(\beta + N + 1 - z)\Gamma(1 + z)\Gamma(\alpha + N + 2 + z), \end{aligned}$$

and C is a complex contour from $-\infty i$ to ∞i which separates the poles of the functions $\Gamma(-z)\Gamma(\beta + N + 1 - z)$ and $\Gamma(1 + z)\Gamma(\alpha + N + 2 + z)$.

Wilson → Racah

- Factorization: If $\alpha+1=-N$ we get

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = \quad R_{N+1}(\lambda(x);-N-1,\beta,\gamma,\delta)(-1)^{n-N-1}$$
$$\times W_{n-N-1}\left(\left(i\left(x+\frac{\gamma+\delta+1}{2}\right)\right)^2; N+\frac{\gamma+\delta+3}{2}, \frac{-\gamma-\delta+1}{2}, \beta+\frac{-\gamma+\delta+1}{2}, \frac{\gamma-\delta+1}{2}\right).$$

- The Δ-Sobolev orthogonality:

$$\langle p,q\rangle_S = \langle p,q\rangle_d + \left\langle (\Delta/\Delta\lambda)^{N+1} p, (\Delta/\Delta\lambda)^{N+1} q \right\rangle_c,$$

with

$$\langle p,q\rangle_d = \sum_{x=0}^{N} p(x)q(x)\frac{(\alpha+1,\beta+\delta+1,\gamma+1,\gamma+\delta+1,(\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1,-\beta+\gamma+1,(\gamma+\delta+1)/2,\delta+1,1)_x},$$
$$\langle p,q\rangle_c = \int_C p(z^2)q(z^2)\nu(zi+i+i(\gamma+\delta+N)/2)\nu(-(zi+i+i(\gamma+\delta+N)/2))dz.$$

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The limits relations between the families

- Racah → Hahn.

$$\lim_{\delta\to\infty} R_n(\lambda(x); -N-1, \beta+\gamma+N+1, \gamma, \delta) = h_n^{\gamma,\beta}(x; N).$$

- Racah → Dual Hahn.

$$\lim_{\beta\to\infty} R_n(\lambda(x); -N-1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).$$

- Hahn → Krawtchouk.

$$\lim_{t\to\infty} h_n^{(1-p)t, pt}(x; N) = K_n(x; p, N).$$

- Dual Hahn → Krawtchouk.

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Orthogonality relations for Meixner polynomials with general parameter

The Meixner polynomials and continuous Hahan polynomials are related through the following limit relation:

$$\lim_{|t|\to\infty}(-i)^n p_n(ix;0,-t/c,t,\beta) = M_n(x;\beta,c), \qquad n=0,1,2,\dots.$$

Proposition: For any $\beta, c \in \mathbb{C}$, $c \notin [0,\infty)$ and $-\beta \notin \mathbb{N}$, the following property of orthogonality for the Meixner polynomials fulfills:

$$\int_C M_n(z;c,\beta) z^m \Gamma(-z)\Gamma(\beta+z)(-c)^z dz = 0,\ 0 \le m < n,\ n=0,1,2,\dots,$$

where C is a complex contour from $-\infty i$ to ∞i separating the increasing poles $\{0,1,2,\dots\}$ from the decreasing poles $\{-\beta,-\beta-1,-\beta-2,\dots\}$.

Some references

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Finally...

Thank for your attention!!