# Old and new results on Sobolev and SemiClassical Orthogonal Polynomials

**Roberto S. Costas-Santos**

www.rscosan.com

joint work with F. J. Marcellán, J. J. Moreno-Balcázar, and J. F. Sánchez-Lara

Work supported by MCeI grant MTM2009-12740-C03-01

Alcalá de Henares, February 28th, 2012

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   - Semiclassical Orthogonal Polynomials
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   - Some identities and results
   - Two examples

# Classical Orthogonal Polynomials

- Let $(P_n)$ be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1, \ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

- COP: Jacobi, Hermite, Laguerre, Bessel.
  Discrete-COP: Hahn, Racah, Meixner, Krawtchouk, Charlier, Askey-Wilson, $q$-Racah, $q$-Hahn, Al Salam Carlitz I, II, etc.

Let $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x).$$

**Favard's theorem**

If $\gamma_n \neq 0 \ \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with $r_n$ a non-vanishing normalization factor.

# Degenerate version of Favard's theorem. Some history

- K. H. Kwon and L. L. Littlejohn, $(L_n^{(-k)})$ orthogonal w.r.t.

$$\langle f, g\rangle = (f(0), f'(0), \dots, f^{(k-1)}(0))A(g(0), g'(0), \dots, g^{(k-1)}(0))^T + \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx.$$

- The Jacobi polynomials $(P_n^{(-1,-1)})$ orthogonal w.r.t.

$$(f, g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x)dx.$$

- These examples suggest that COP with non-classical parameters can be provided with a orthogonality of Sobolev-type.
- Furthermore F. Marcellan and J.J. Moreno-Balcázar pointed out that a Sobolev-Askey tableau should be established.

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- These examples suggest that COP with non-classical parameters can be provided with a orthogonality of Sobolev-type.
- Furthermore F. Marcellan and J.J. Moreno-Balcázar pointed out that a Sobolev-Askey tableau should be established.

- Let $\mathscr{T}_1 : \mathbb{P}[x] \to \mathbb{P}[x]$ be a linear operator such that
  - $\deg \mathscr{T}_1(p) = \deg p - 1$
  - The **monic** polynomials sequence $(p_{n,1})$ defined by

$$p_{n,1} := \mathrm{const.}\mathscr{T}_1(p_{n+1}),$$

fulfill the TTRR

$$xp_{n,1}(x) = p_{n+1,1}(x) + \beta_{n,1}p_{n,1}(x) + \gamma_{n,1}p_{n-1,1}(x)$$

so that there exists $\lambda : \{\gamma_{n,1} = 0\} \to \{\gamma_n = 0\}$ strictly increasing with $\lambda(n) > n$.

**Remark**

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# The iterative process

1. $p_{n,k} := \mathrm{const.}\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \mathrm{const.}\mathscr{T}^{(k)}(p_{n+k})$.
2. $xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k}p_{n,k}(x) + \gamma_{n,k}p_{n-1,k}(x)$
3. $\mathscr{L}_k(p_{m,k}p_{n,k}) = 0$ for $n \neq m$.
4. The first $n$ such that $\gamma_{n,k} = 0$ (if it exists) verifies $n < N - k$.

**Theorem:**

Suppose that only $\gamma_N = 0$, then $(p_n)$ is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Notice $\gamma_{n,N} \neq 0$ for all $n \in \mathbb{N}$.

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Notice $\gamma_{n,N} \neq 0$ for all $n \in \mathbb{N}$.

# Degenerate version of Favard's theorem

**Corollary**

If $\Lambda = \{n : \gamma_n = 0\}$, then $(p_n)$ is a MOPS with respect to

$$\langle f, g \rangle = \mathscr{L}_0(fg) + \sum_{j \in \mathscr{A}} \mathscr{L}_j(\mathscr{T}^{(j)}(f)\mathscr{T}^{(j)}(g)),$$

being $\mathscr{A} = \{N_0, N_1, \dots\}$ with $N_{j+1} = N_j + \min\{n : \gamma_{n,N_j} = 0\}$.

1. Among all the possible choices the linear operator $\mathscr{T}$ can be chosen as The "Associating operator"

$$\mathscr{T}(p)(x) = \mathscr{L}_0\left(\frac{p(x)-p(t)}{x-t}\right).$$

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The monic ones are $p_n(x; a, b, c, d; q) \equiv p_n(x)$

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

with

$$\frac{\gamma_n}{1-q^n} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{4(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$$

Case $abcd \in \{q^{-k} : k \in \mathbb{N}_0\}$ are not considered since they are not normal.
They are symmetric with respect to any rearrangement of the parameters $a$, $b$, $c$, $d$.

$$\{n \in \mathbb{N} : \gamma_n = 0\} \neq \emptyset \iff ab, ac, \dots, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$$

$$\iff \text{they are } q\text{-Racah.}$$

# Orthogonality of AW polynomials for $|q| < 1$

$$\int_C p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n\delta_{n,m}$$

where

- $W$ is analytic in $\mathbb{C}$ except at the poles 0,

$$aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0 \quad \text{(the convergent poles)}$$

$$(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0 \quad \text{(the divergent poles)}$$

- $C$ is the unit circle deformed to separate the convergent form the divergent poles.

# The 3 key cases

- Case I: $a^2 = q^{-N+1}$ and

$$b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case II: $ab = q^{-N+1}$ and

$$a^2, b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case III: $ab = q^{-N+1}$, $a^2 = q^{-M}$ with $M \in \{0, 1, \ldots, N-2\}$ and

$$b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

# Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1$: By using the identity

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

- $|q| = 1$: If $q = \exp(2M\pi/N \text{ I})$, then $\gamma_{jN} = 0, j \in \mathbb{N}$.

  - Spiridonov and Zhedanov found $\mathscr{L}_0$
  - For $n > N$

$$\mathscr{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q).$$

  - $\mathscr{L}_j(p(\cdot)) = \mathscr{L}_0(p((-1)^M \cdot))$

- For the rest of the values of $q$ the result keeps unknown.

- Let $(P_n)$ be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of quasi-orthogonality of order $\delta$

$$\langle \mathbf{u}, P_n P_m \rangle = 0 \; |n-m| > \delta, \; \exists r \geq \delta \; : \; \langle \mathbf{u}, P_r P_{r-\delta} \rangle \neq 0.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1.$$

- A 'recurrence relation':

$$P_n = M_{r-1} Z_{n-r+1} + N_{r-2} Z_{n-r}.$$

- There is not a general classification.

# Some definitions

## Admisibility

The pair of polynomials $(\phi, \psi)$ is an admissible pair if one of the following conditions is satisfied:

- $\deg \psi \neq \deg \phi - 1$,
- $\deg \psi = \deg \phi - 1$, with $a_p + q^{-1}[n]^* b_t \neq 0$, where $a_p$ and $b_t$ are the leading coefficients of $\psi$ and $\phi$, respectively.

## Order and class of a linear functional

$\sigma := \max\{\deg \phi - 2, \deg \psi - 1\}$. The class of $\mathbf{u}$ is the min. order from among all the adm. pairs.

## The sequence $(\phi_k)$ and $(\mathbf{u}_k)$

Given a semiclassical functional $\mathbf{u}$ satisfying PE, for $k \in \mathbb{Z}$ we define the $\mathbf{u}_k$ as: $\mathbf{u}_k = \mathscr{E}^+(\phi_{k-1}\mathbf{u}_{k-1})$, $\mathbf{u}_0 = \mathbf{u}$, $\phi_0 = \phi$, where $\phi_k$ is a multiple of $\phi_{k-1}$.

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## Theorem 1

Let $(p_n)$ be a sequence of monic OP w.r.t. $\mathbf{u}$, and $\phi$ pol. of degree $t$. The following statements are equivalent:

1. There exist three non–negative integers, $\sigma$, $p$, and $r$, with $p \geq 1$, $r \geq \sigma + t + 1$, and $\sigma = \max\{t-2, p-1\}$, s.t.

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} p_\nu(z) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} p_\nu^{[1]}(z),$$

   where $p_n^{[1]}(z) := [n+1]^{-1}(\mathscr{D} p_{n+1})(z)$.

2. There exists a polynomial $\psi$, with $\deg \psi = p \geq 1$, such that

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u},$$

   where the pair $(\phi, \psi)$ is an admissible pair.

- Let $\mathbf{u}$ be a semiclassical functional of order $\sigma$, and let $(p_n)$ be the monic OP sequence w.r.t. $\mathbf{u}$.
- Consider the $\mathscr{D}$–Sobolev inner product defined by

$$\langle p, r\rangle_S = \langle \mathbf{u}, p\, r\rangle + \lambda\langle \mathbf{u}, \mathscr{D}p\,\mathscr{D}r\rangle, \qquad \lambda \geq 0.$$

- Let $(Q_n^{(\lambda)})$ be the OP sequence associated with the ($\mathscr{D}$–Sobolev) inner product $\langle\cdot,\cdot\rangle_S$ which we call semiclassical Sobolev orthogonal polynomial.

**Proposition**

For $n \geq \sigma_{-1} + H^*$,

$$\sum_{\nu=n-\sigma_{-1}}^{n+\sigma_{-1}} \xi^*_{n,\nu} p_\nu^{\{-1\}}(z) = Q^{(\lambda)}_{n+\sigma_{-1}}(z) + \sum_{\nu=n-\sigma_{-1}-H^*}^{n-1+\sigma_{-1}} \theta_{n,\nu}\, Q_\nu^{(\lambda)}(z),$$

where $H^* := \max\{t_{-1}, \sigma_{-1}\}$.

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where $H^* := \max\{t_{-1}, \sigma_{-1}\}$.

# two identities and a Theorem

## Identity 1

Let $\mathscr{J}$ be the linear operator

$$\mathscr{J} := (\mathscr{E}^{-}\widetilde{\phi})\mathscr{I} + \frac{\lambda}{q}(\mathscr{D}^{*}\widetilde{\phi} - \widetilde{\psi})\mathscr{D}^{*} - \lambda(\mathscr{E}^{-}\widetilde{\phi})\mathscr{D}\,\mathscr{D}^{*},$$

where $\mathscr{I}$ is the identity operator. Then,

$$\langle(\mathscr{E}^{-}\widetilde{\phi})p, r\rangle_S = \langle\mathbf{u}, p\,\mathscr{J}\,r\rangle, \quad p,\ r \in \mathbb{P}.$$

## Identity 2

$$\langle(\widetilde{\psi} - \mathscr{D}^{*}\widetilde{\phi})p, r\rangle_S = \langle\mathscr{D}^{*}\mathbf{u}, p\,\mathscr{J}\,r\rangle, \quad p,\ r \in \mathbb{P}.$$

## Theorem

$$\langle\mathscr{J}\,p, r\rangle_S = \langle p, \mathscr{J}\,r\rangle_S \qquad p,\ r \in \mathbb{P}.$$

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$$\langle \mathscr{J}\,p, r\rangle_S = \langle p, \mathscr{J}\,r\rangle_S \qquad p,\, r \in \mathbb{P}.$$

## Corollary

The following relations hold

$$(\mathscr{E}^{-}\widetilde{\phi})(z)p_n(z) = \sum_{\nu=n-H}^{\nu=n+\deg\widetilde{\phi}} \mu_{n,\nu} Q_\nu^{(\lambda)}(z), \quad n \geq H,$$

$$\mathscr{J}\, Q_n^{(\lambda)}(z) = \sum_{\nu=n-\deg\widetilde{\phi}}^{n+H} \vartheta_{n,\nu} p_\nu(z), \quad n \geq \deg\widetilde{\phi},$$

$$\mathscr{J}\, Q_n^{(\lambda)}(z) = \sum_{\nu=n-H}^{n+H} \varpi_{n,\nu} Q_\nu^{(\lambda)}(z), \quad n \geq H,$$

where $H := \max\{\deg\widetilde{\psi} - 1, \deg\widetilde{\phi}\}$.

# A $q$ example: $q$-Freud type polynomials

- The monic $q$–Freud polynomials, $(P_n)$, satisfies the relation

$$(\mathscr{D}P_n)(x(s)) = [n]P_{n-1}(x(s)) + a_nP_{n-3}(x(s)), \qquad n \geq 0,$$

where $x(s) = q^s$, with $0 < q < 1$, $\mathscr{D} = \mathscr{D}_q$.

- $\phi(x) = 1$, $t = 0$, and $(P_n)$ is orth. w.r.t. $\mathbf{u}^{qF}$ of class $\sigma = 2$.
- $\mathscr{D}(\mathbf{u}^{qF}) = \psi\mathbf{u}^{qF}$, $\quad \mathscr{D}^*((1 + x(q-1)\psi)\mathbf{u}^{qF}) = q\psi\mathbf{u}^{qF}$, $\deg\psi = 3$.
- $\mathbf{u}^{qF}$ has the following integral representation:

$$\left\langle \mathbf{u}^{qF}, P \right\rangle = \int_{-1}^{1} P(x)\,\frac{1}{((q-1)K(q)q^{-3}q^{4x};q^4)_\infty} d_q(x),$$

# Cont.

$$\mathscr{J}^{qF} = (1-(q-1)K(q)q^{-7}x^4)\mathscr{I} + \frac{\lambda}{q}K(q)q^{-6}x^3\mathscr{D}_{1/q} - \lambda(1-(q-1)K(q)q^{-7}x^4)\mathscr{D}_q\mathscr{D}_{1/q}.$$

- $H = \deg \widetilde{\phi} = 4$ and

$$(1-(q-1)K(q)q^{-7}x^4)P_n(x) = \sum_{\nu=n-4}^{n+4} \mu_{n,\nu}^{qF} Q_\nu^{qF}(x),$$

$$\mathscr{J}^{qF} Q_n^{qF}(x) = \sum_{\nu=n-4}^{n+4} \vartheta_{n,\nu}^{qF} P_\nu(x),$$

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# A 1–singular semiclassical polynomials of class 1

- This family, $(S_n)$, was studied by J.C. Medem and it is orthogonal w.r.t. $\mathbf{w}$.
- Distributional equation:

$$\mathscr{D}(x^3\mathbf{w}) = (-x^2+4)\mathbf{w}.$$

- $\mathscr{D} = \mathscr{D}^* = \dfrac{d}{dx}$, $t=3$, $p=2$, so $\sigma = 1$; with initial condition $(\mathbf{w})_1 = \langle \mathbf{w}, x \rangle = 0$.
- $\mathbf{w}$ is 1–singular.
- $\mathbf{w}$ is the symmetrized of $\mathbf{b}^{(-\frac{5}{2})}$.
- $\mathbf{w}$ has the following integral representation:

$$\langle \mathbf{w}, P \rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} P(z) z^{-4} e^{-\frac{2}{z^2}}\, dz.$$

# Cont.

$$\mathscr{J}^{S} = x^3\mathscr{I} + \lambda(4x^2 - 4)\frac{d}{dx} - \lambda x^3\frac{d^2}{dx^2}.$$

- $H = \deg \widetilde{\phi} = 3$ and

$$x^3 S_n(x) = Q^S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \mu^S_{n,\nu} Q^S_\nu(x),$$

$$\mathscr{J}^{S} Q^S_n(x) = S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \vartheta^S_{n,\nu} S_\nu(x),$$

$$\mathscr{J}^{S} Q^S_n(x) = Q^S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \varpi^S_{n,\nu} Q^S_\nu(x).$$

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