CARMA SEMINAR 16TH FEB 2016

BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

Roberto Costas-Santos

Universidad de Alcalá

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WHERE IS ALCALÁ DE HENARES (UNESCO'S WORLD HERITAGE SITES)

THE CITY WHERE **CERVANTES** WAS BORN.. HE WROTE **THE QUIJOTE**

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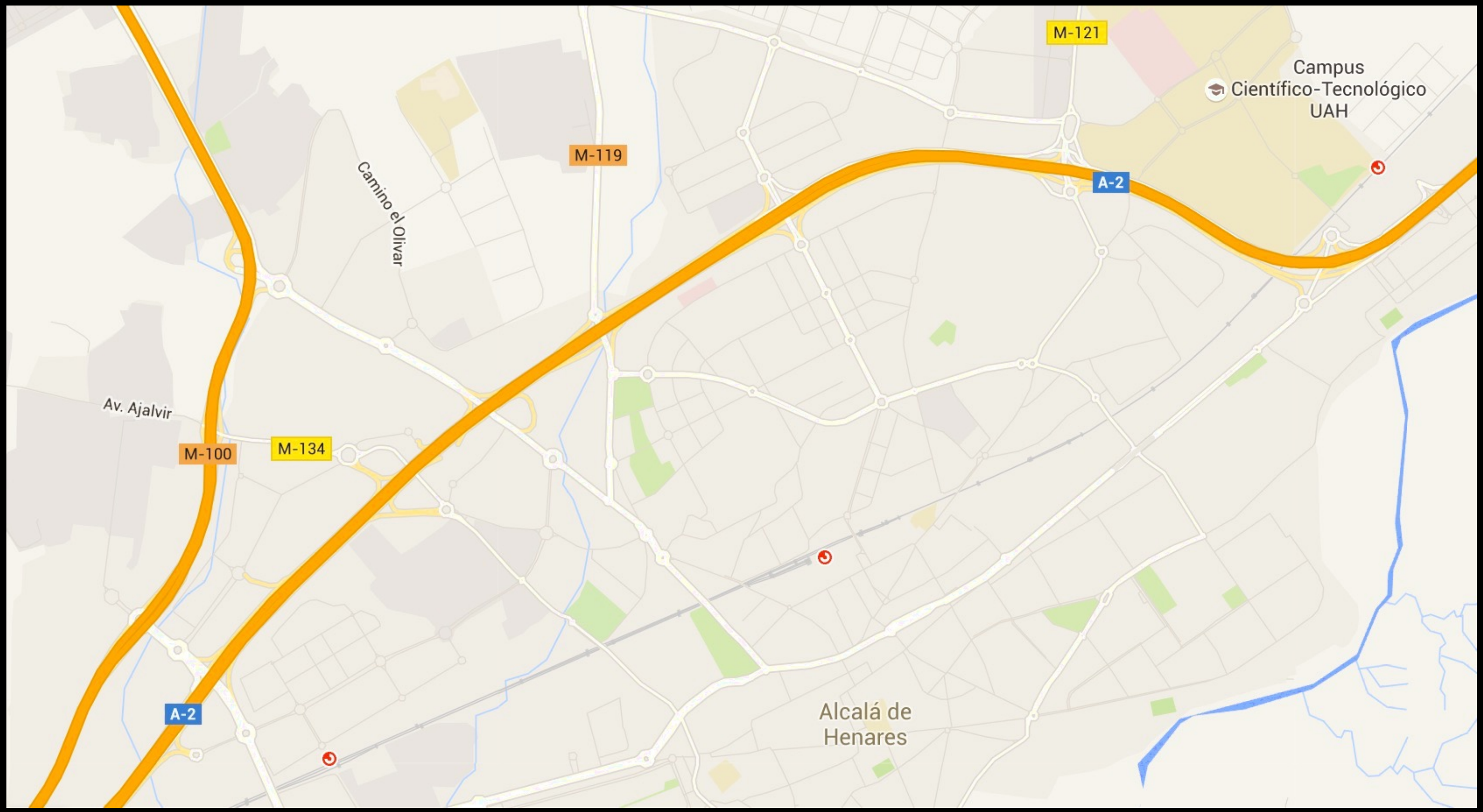
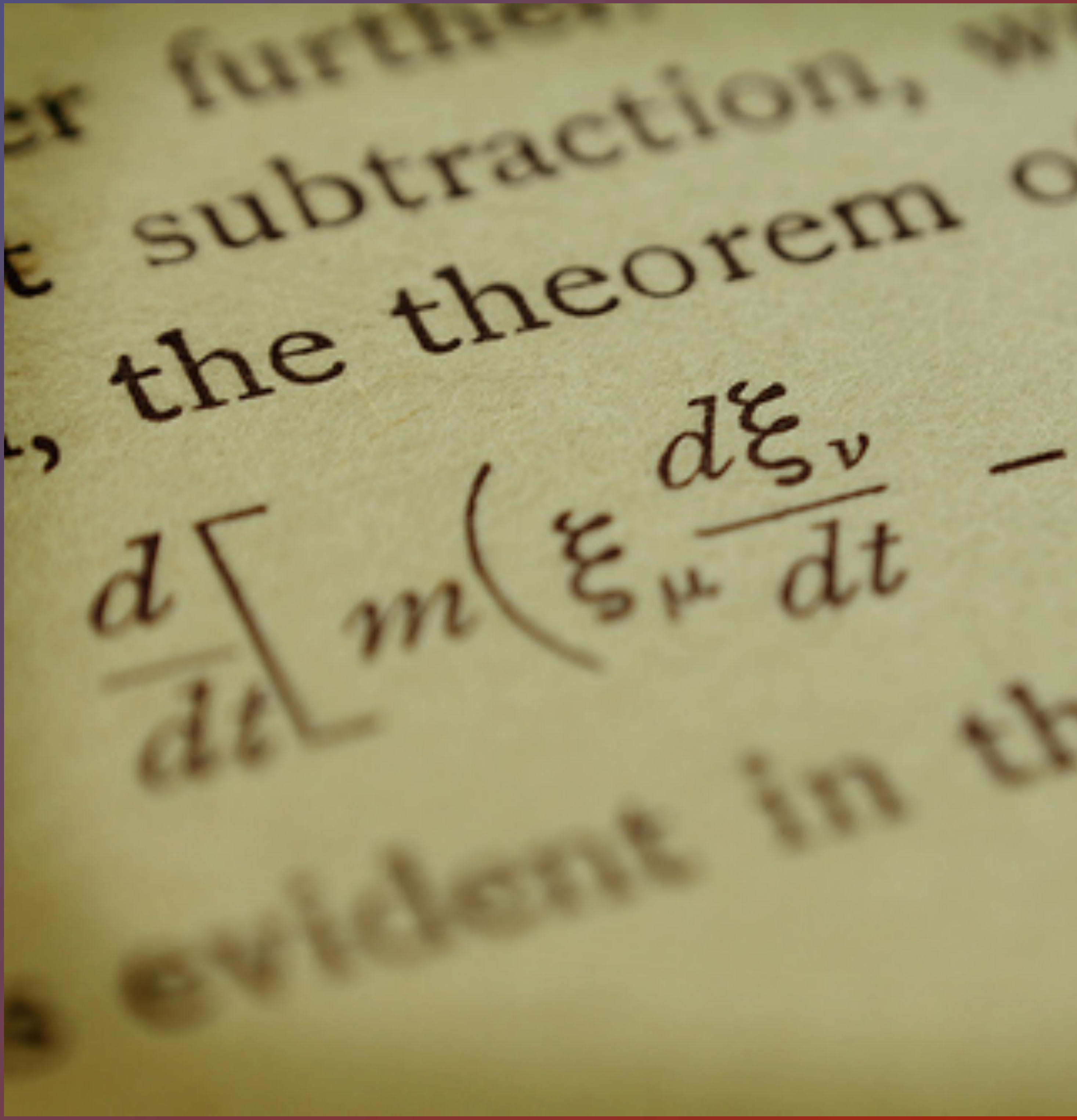


OUTLINE

* BASICS
* COP
* THE COP SCHEME
* Q-COP
* THE Q-COP SCHEME
* THE ASC POL.



SOME BASICS OF CLASSICAL OP

- Let **u** be a linear functional.
- If **u** fulfills the distributional equation

$$\mathcal{D}(\phi\mathbf{u}) = \psi\mathbf{u}, \quad \deg\psi \le 1,\ \deg\phi \le 2$$

- Property of orthogonality: $\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}$
  Three-term recurrence relation:

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

- Integral representation with a weight function

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}. \qquad d\mu(z) = \omega(z)\, dz$$

Characterization Theorems. The continuous version

Let (P_n) be an OPS with respect to ω. The following statements are equivalent:

1. P_n is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. (P'_{n+1}) is a OPS.
3. $(P^{(k)}_{n+k})$ is a OPS for any integer k.
4. (First structure relation)

$$\phi(x)P'_n(x) = \widehat{\alpha}_n P_{n+1}(x) + \widehat{\beta}_n P_n(x) + \widehat{\gamma}_n P_{n-1}(x).$$

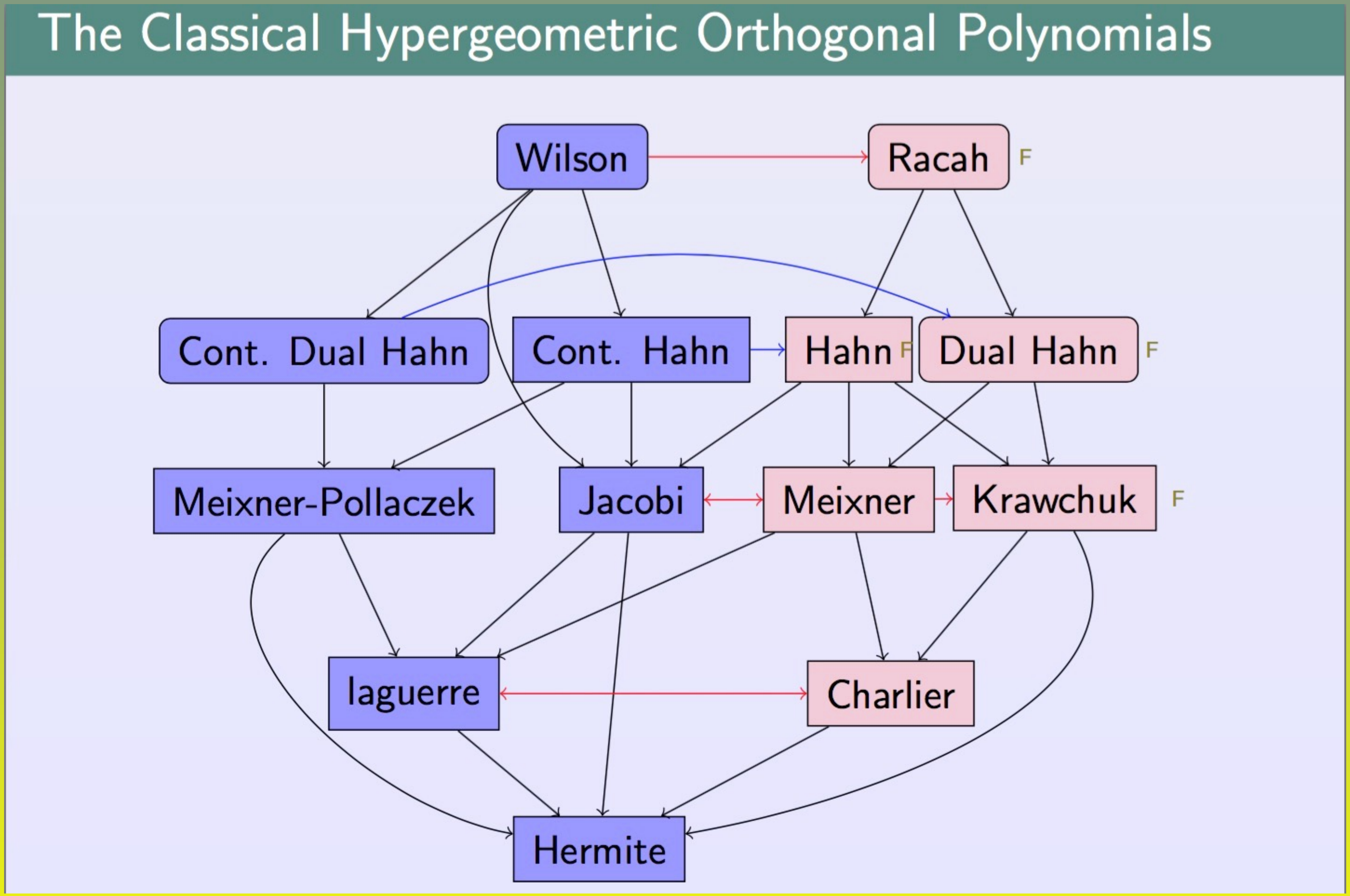
5. (Second structure relation)

$$P_n(x) = \widetilde{\alpha}_n P'_{n+1}(x) + \widetilde{\beta}_n P'_n(x) + \widetilde{\gamma}_n P'_{n-1}(x).$$

6. (Eigenfunctions of SODE)

$$\phi(x)P''(x) + \psi(x)P'(x) + \lambda P(x) = 0.$$

CLASSICAL OP



Representation of the Classical OP

Hypergeometric and basic hypergeometric representations

The continuous and discrete COP can be written in terms of

$$ {}_rF_s\left(\begin{array}{c} a_1,\ a_2,\ \ldots,a_r \\ b_1,\ b_2,\ \ldots,b_s \end{array}\middle| z\right) = \sum_{k\geq 0} \frac{(a_1)_k(a_2)_k\ldots(a_r)_k}{(b_1)_k(b_2)_k\ldots(b_s)_k}\frac{z^k}{k!}. $$

The q-discrete COP can be written in terms of

$$ {}_r\varphi_s\left(\begin{array}{c} a_1,\ \ldots,a_r \\ b_1,\ \ldots,b_s \end{array}\middle| z\right) = \sum_{k\geq 0} \frac{(a_1;q)_k\ldots(a_r;q)_k}{(b_1;q)_k\ldots(b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}. $$

$$ (a)_k = a(a+1)\cdots(a+k-1) $$

$$ (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) $$

Laguerre and Jacobi Polynomials

$$L_n^{\alpha}(x) \quad = \quad \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \, {}_1\mathrm{F}_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right),$$

$$P_n^{\alpha,\beta}(x) \quad = \quad \frac{2^n (\alpha+1)_n}{(n+\alpha+\beta+1)_n} \, {}_2\mathrm{F}_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right).$$

α_n	1	1	1
β_n	0	$2n+\alpha+1$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}$
γ_n	$\frac{n}{2}$	$n(n+\alpha)$	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$

DEGENERATE FAVARD'S RESULT

DEGENERATE FAVARD'S RESULT

The Favard's theorem

Let $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x).$$

Favard's theorem

If $\gamma_n \neq 0 \; \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with r_n a non-vanishing normalization factor.

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Degenerate version of Favard's theorem

Theorem

If there exists N so that $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \sum_{j\in\mathscr{A}} \mathscr{L}_1(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

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Journal of Computational and Applied Mathematics 225 (2009) 440–451

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Extensions of discrete classical orthogonal polynomials beyond the orthogonality

R.S. Costas-Santos[a,*], J.F. Sánchez-Lara[b]

[a] Department of Mathematics, University of California, South Hall, Room 6607 Santa Barbara, CA 93106, USA

[b] Universidad Politécnica de Madrid, Escuela Técnica Superior de Arquitectura, Departamento de Matemática Aplicada, Avda Juan de Herrera, 4. 28040 Madrid, Spain

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Journal of Approximation Theory

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Full length article

Orthogonality of q-polynomials for non-standard parameters

R.S. Costas-Santos [a,*], J.F. Sánchez-Lara [b]

Q-COP = Q-POLYNOMIALS

- The function $\omega(s)$ fulfills a Pearson-type difference eq.:

$$\phi(s+1)\omega(s+1) - \phi(s)\omega(s) = \big(x(s+1/2) - x(s-1/2)\big)\psi(s)$$

- The *q*-polynomials satisfy, in general, a property of orthogonality

$$\langle \mathbf{u}, P \rangle = \int_a^b P(z)\omega(z)\, d_q z$$

The *q*-integral is defined by

$$\int_0^z f(t)\, d_q t := z(1-q)\sum_{n=0}^{\infty} f(q^n z) q^n, \quad 0 < q < 1.$$

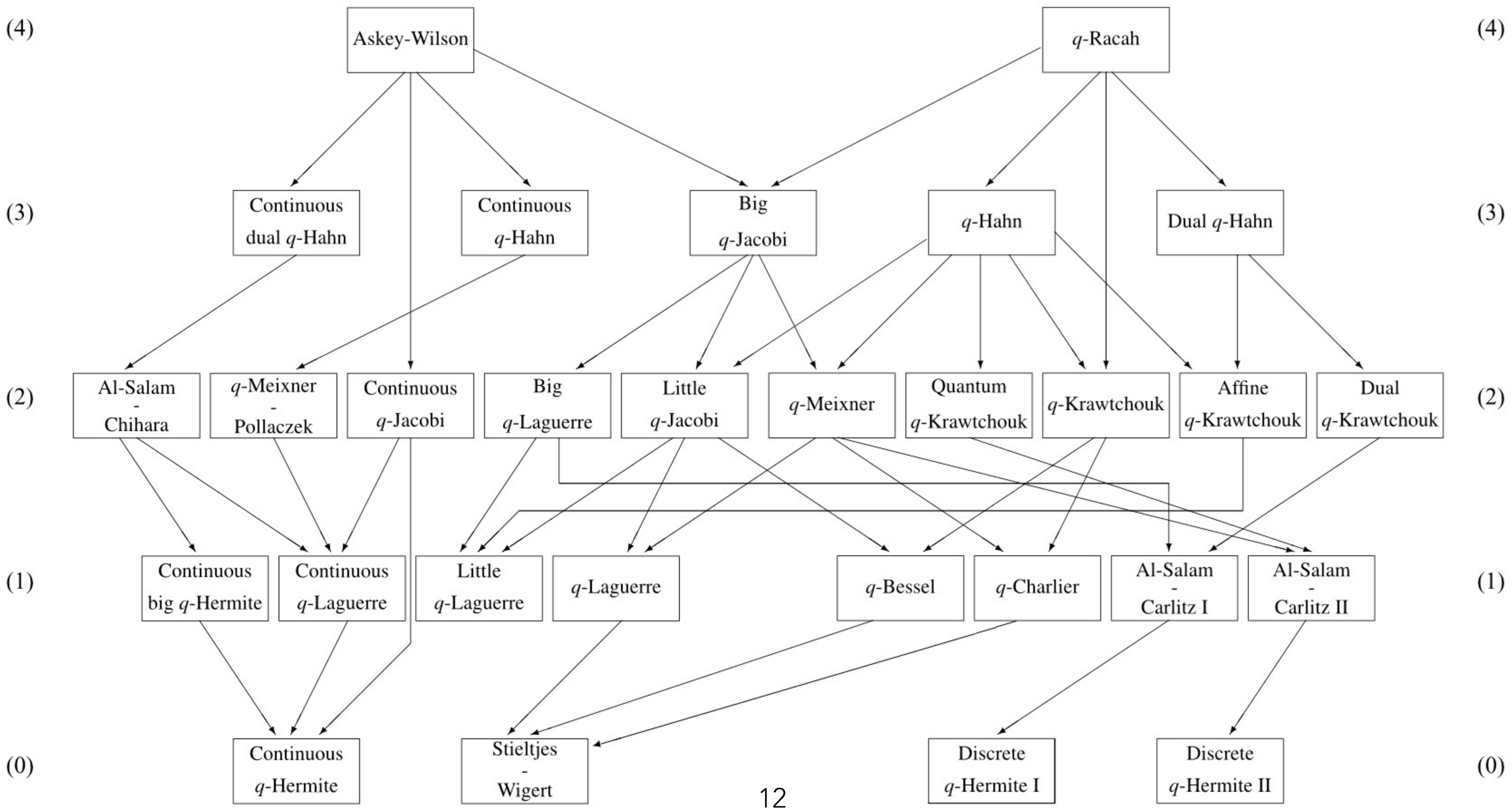
$$\int_a^b f(t)\, d_q t = b(1-q)\sum_{n=0}^{\infty} f(bq^n)q^n - a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n,$$

SCHEME

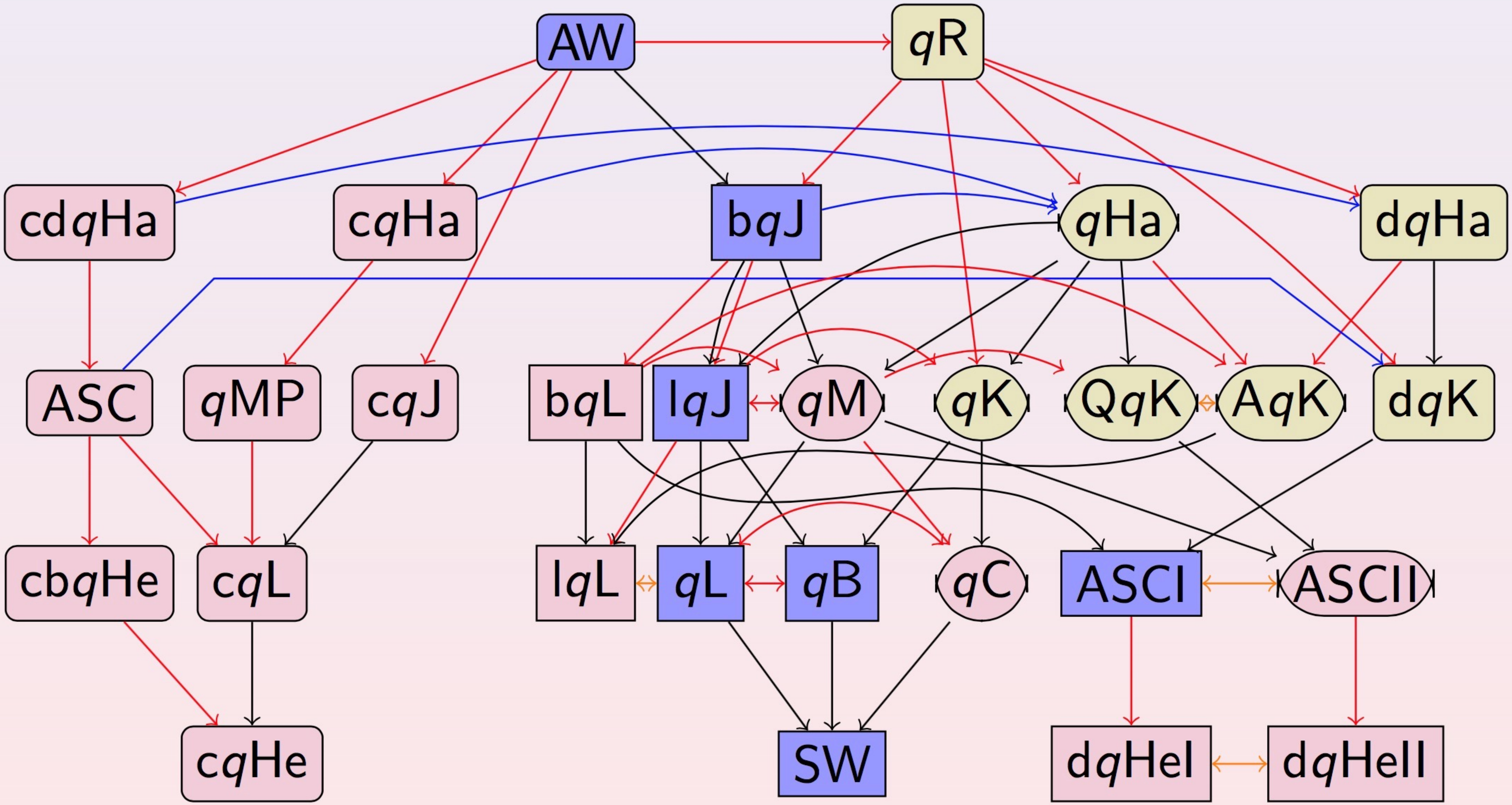
OF

BASIC HYPERGEOMETRIC

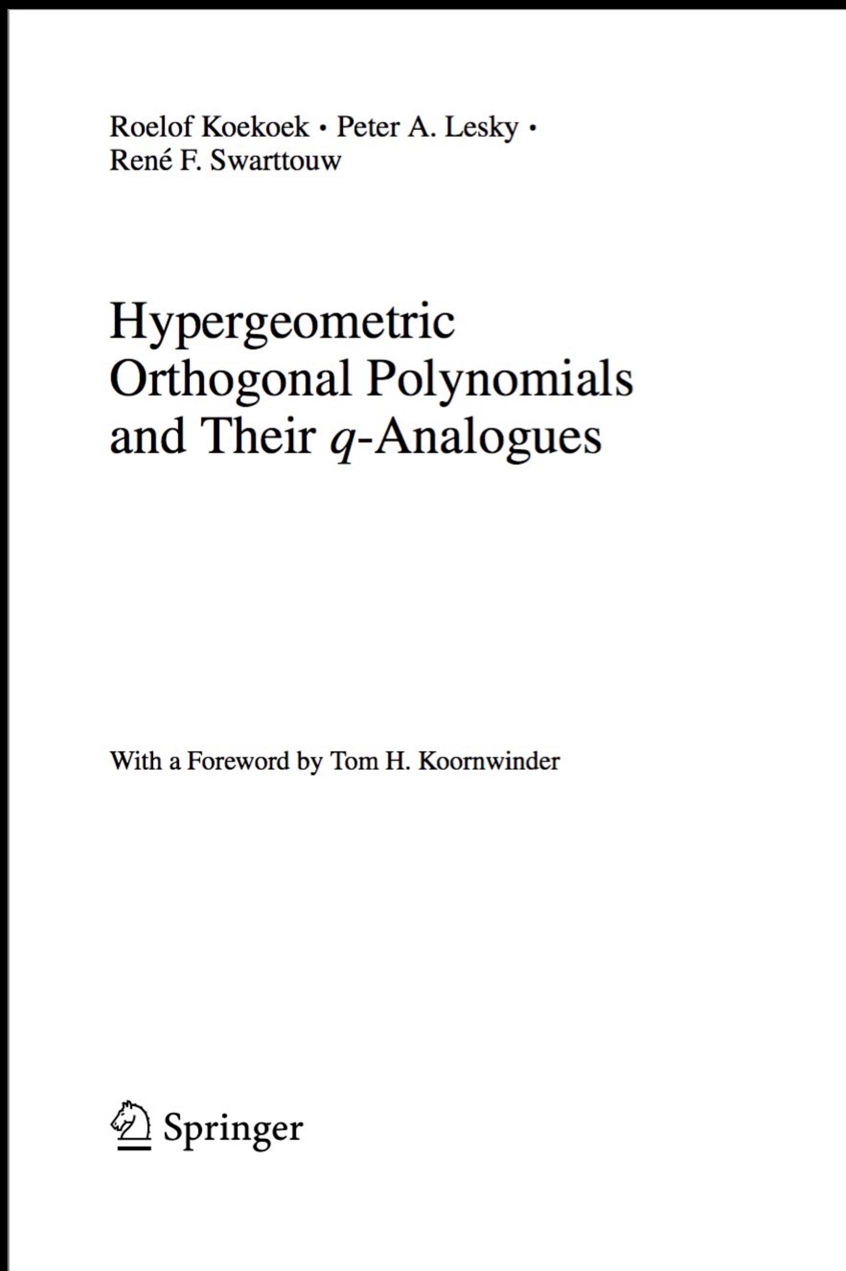
ORTHOGONAL POLYNOMIALS



THE BASIC HYPERGEOMETRIC OP



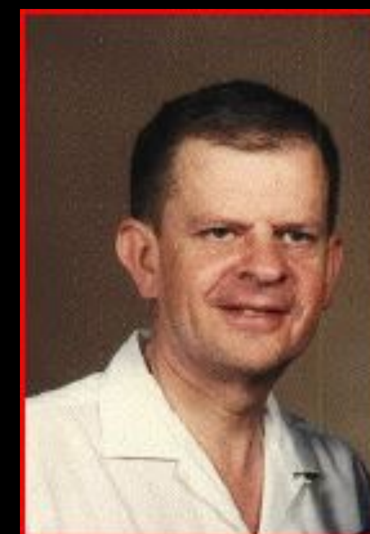
A GOOD REFERENCE ABOUT Q-COP


Roelof Koekoek • Peter A. Lesky •
René F. Swarttouw

Hypergeometric
Orthogonal Polynomials
and Their *q*-Analogues

With a Foreword by Tom H. Koornwinder

Springer



AND SOME PICTURES

DEGENERATE VERSION OF FAVARD'S RESULT IN THE 'Q-WORLD'

The Askey-Wilson polynomials case

$$\frac{\gamma_n}{1-q^n} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{4(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$$

THE SUPPORT OF THE MEASURE AND THE JACOBI MATRIX

Taking into account the TTRR

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

one constructs the Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The spectrum of the N-by-N truncated Jacobi matrix are the zeros of $P_N(x)$ for all N.

THE AL-SALAM-CARLITZ POLYNOMIALS

In this case $\phi(x) = (x-1)(x-a)$

For $a, q \in \mathbb{C},\ a \neq 1,\ 0 < |q| < 1$

14.24 Al-Salam-Carlitz I

Basic Hypergeometric Representation

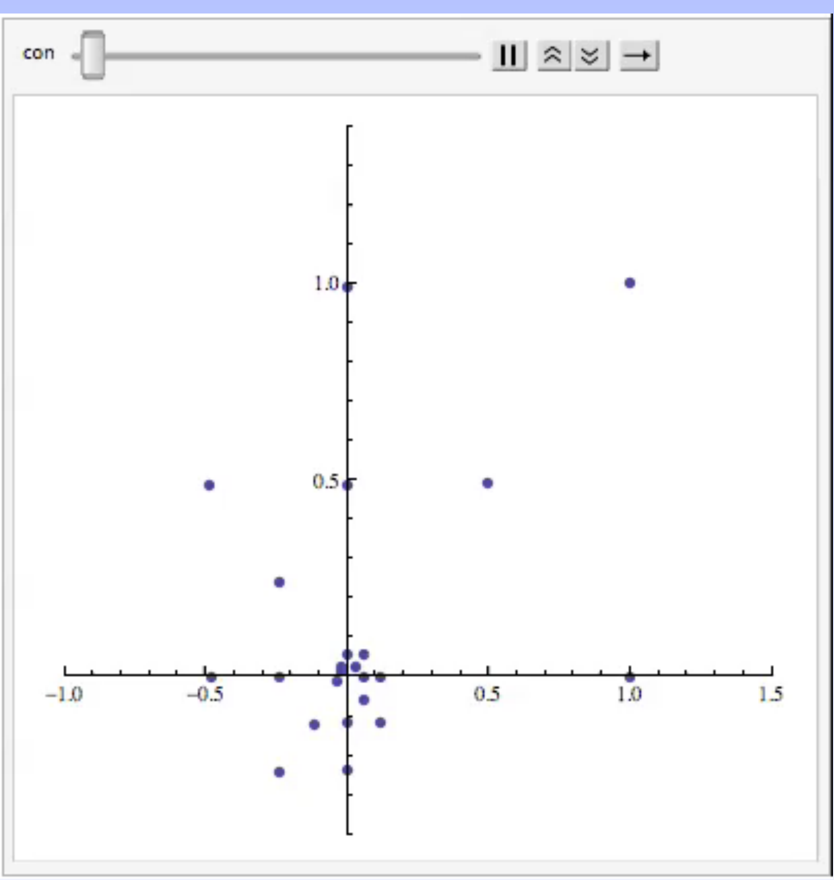
$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, \frac{qx}{a}\right).$$

Orthogonality Relation

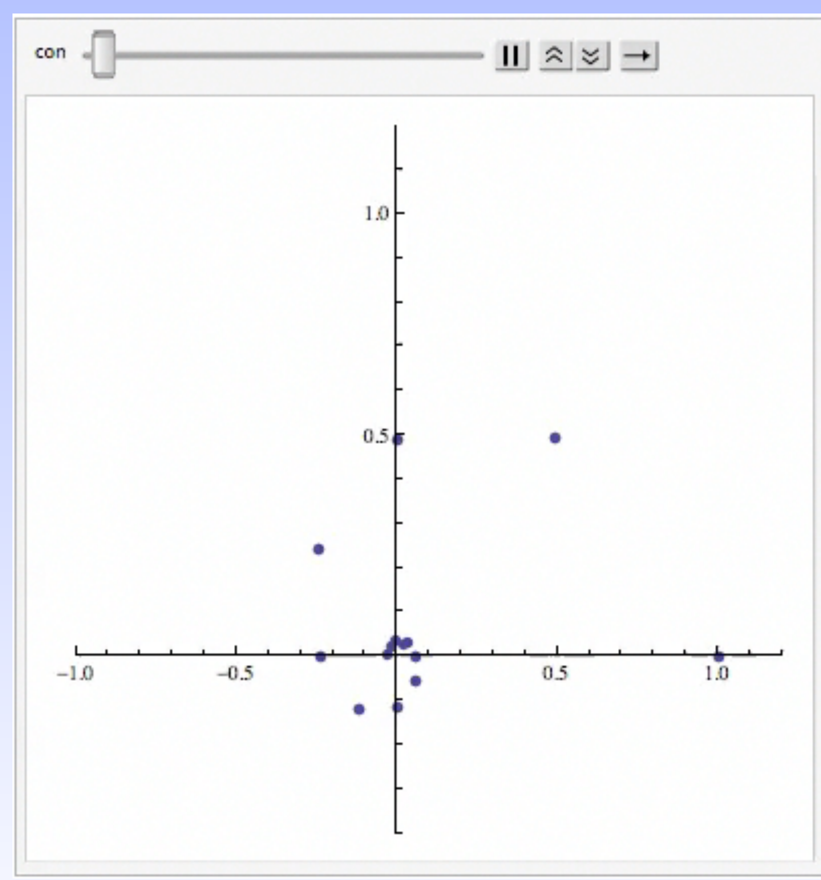
$$\int_a^1 (qx, a^{-1}qx;q)_\infty U_m^{(a)}(x;q) U_n^{(a)}(x;q)\, d_q x$$
$$= (-a)^n (1-q)(q;q)_n (q, a, a^{-1}q;q)_\infty q^{\binom{n}{2}}\, \delta_{mn}, \quad a < 0.$$

Recurrence Relation

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^n U_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q).$$



$a = 1 + I,\ q = 0.7\exp(\pi I/4)$



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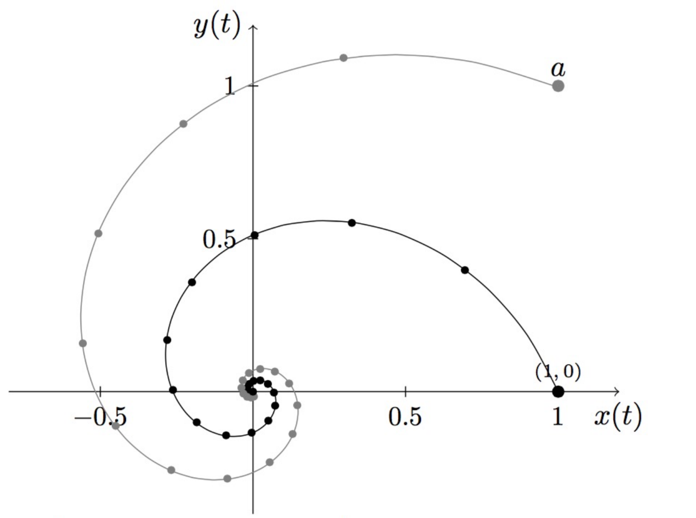
AL-SALAM-CARLITZ POLYNOMIALS. A GENERAL STUDY



The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 0.8\exp(\pi i/6)$.

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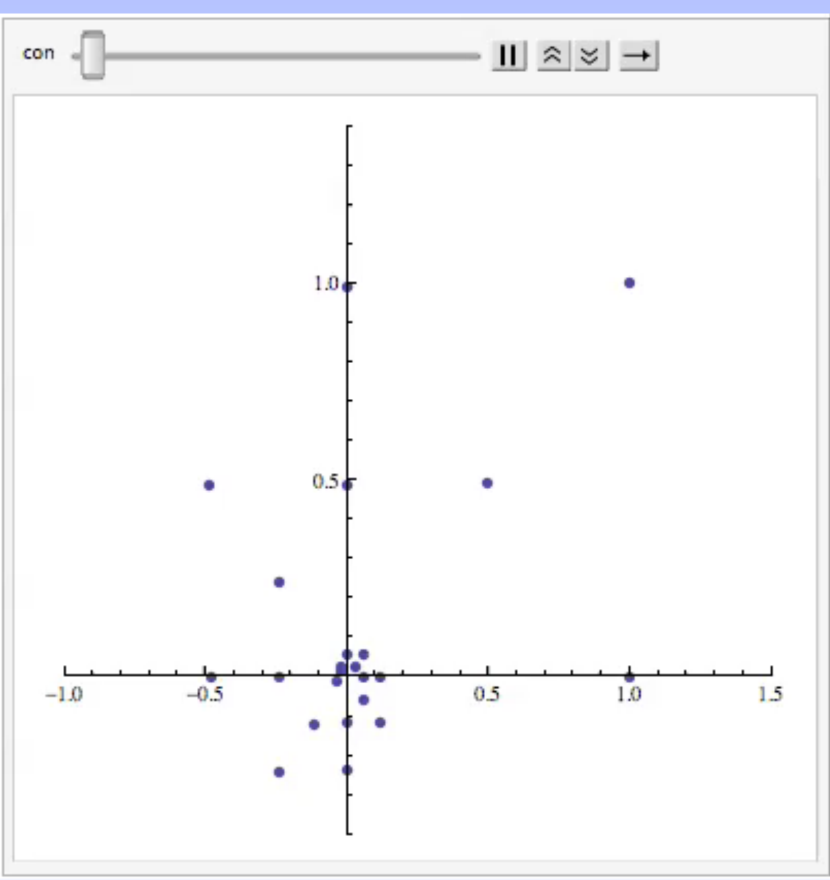

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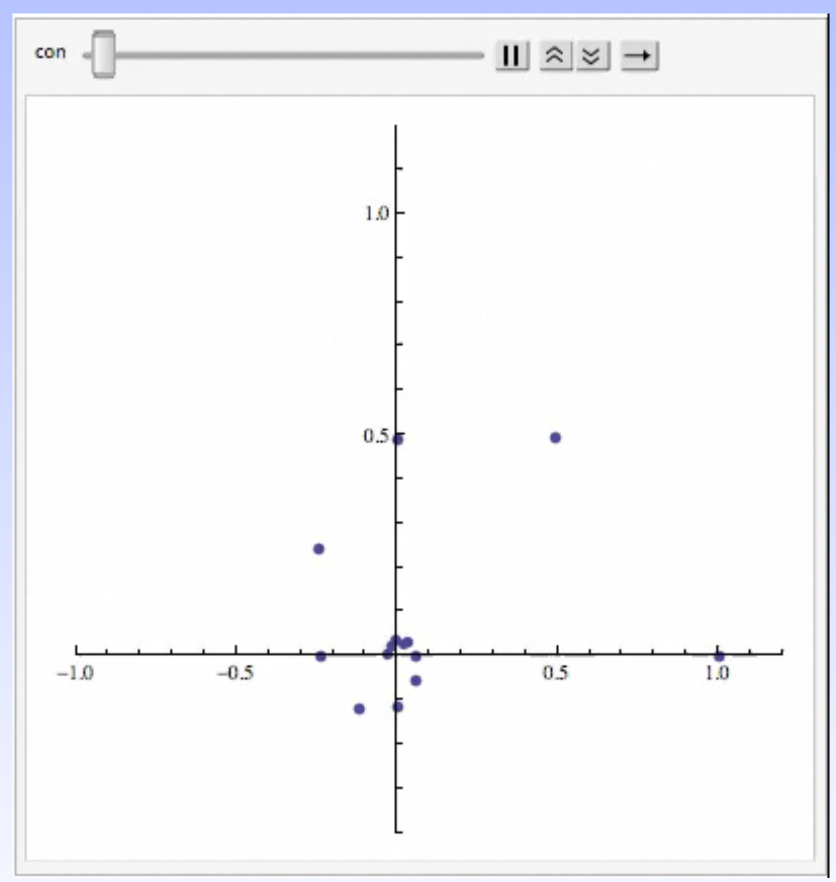

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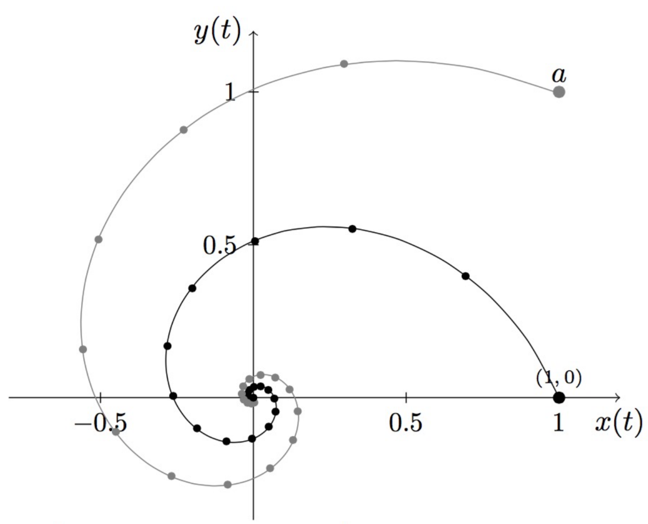

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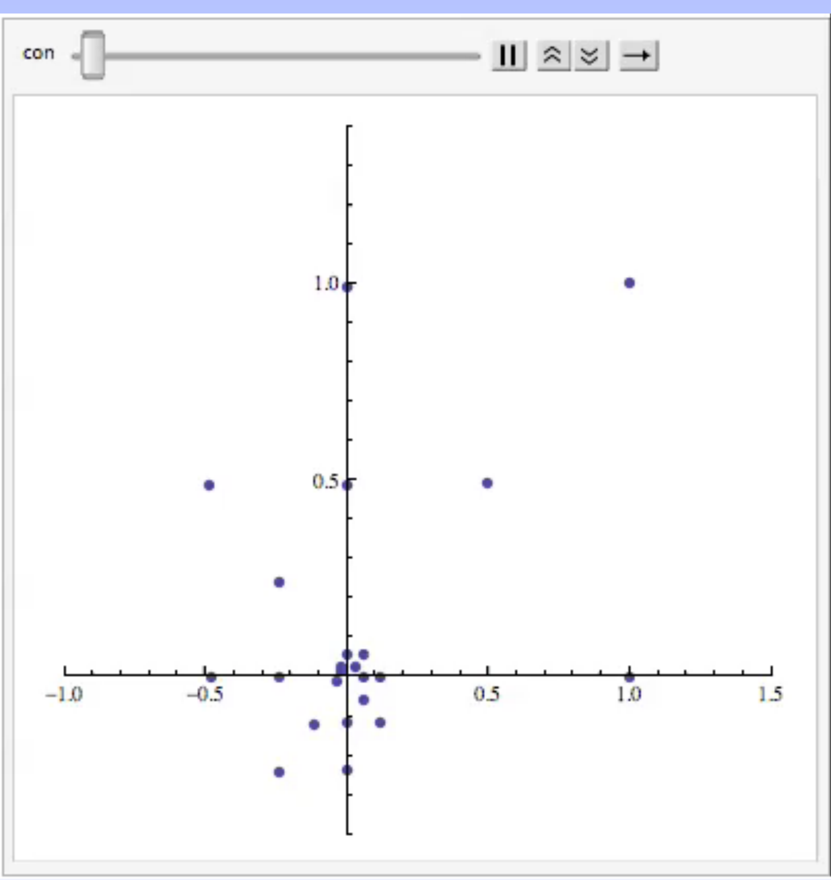

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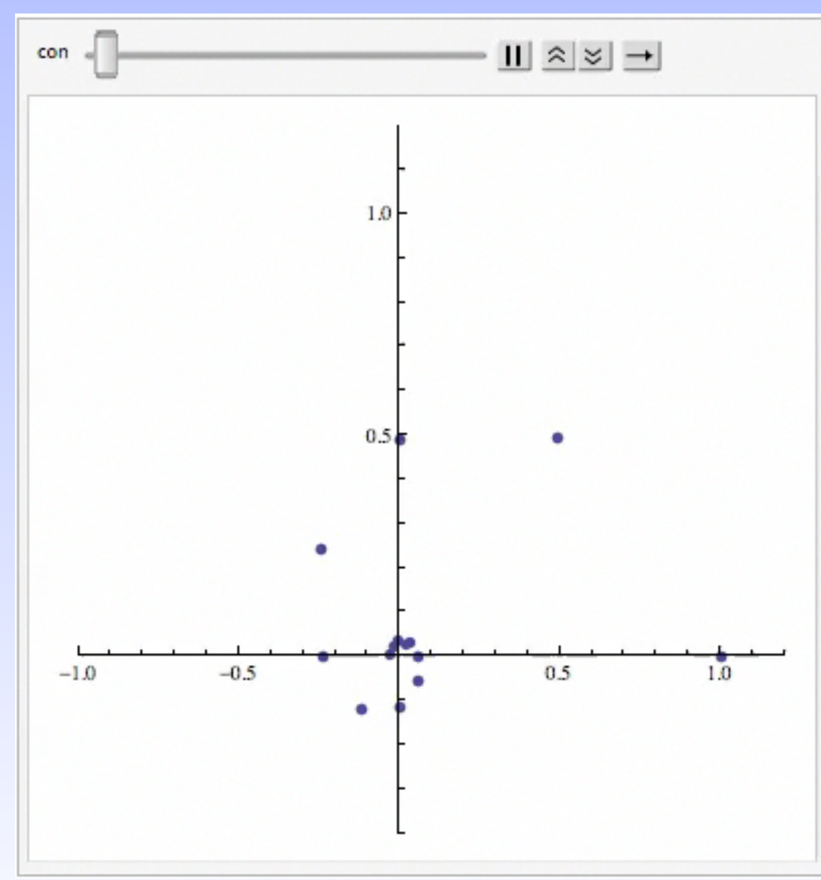

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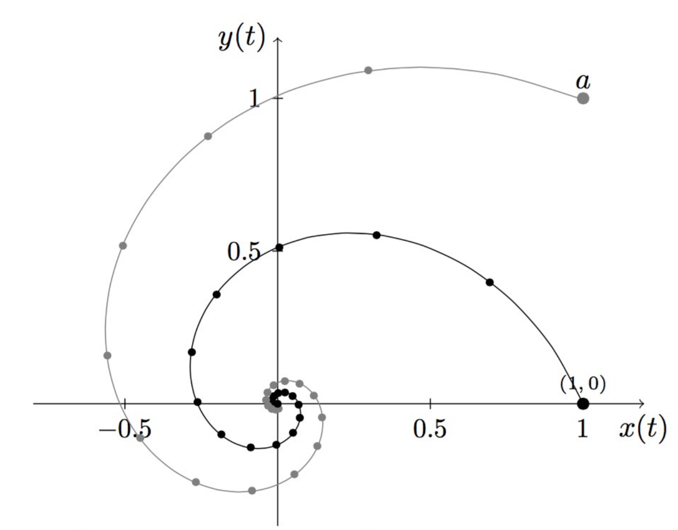

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SOME REFERENCES

Some References

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obrigado
Dank U
Merci
mahalo
Köszi
спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos