

# The orthogonality relations of the Al-Salam-Carlitz polynomials for general parameters

Joint work with Howard S. Cohl, and Wenqing Xu.

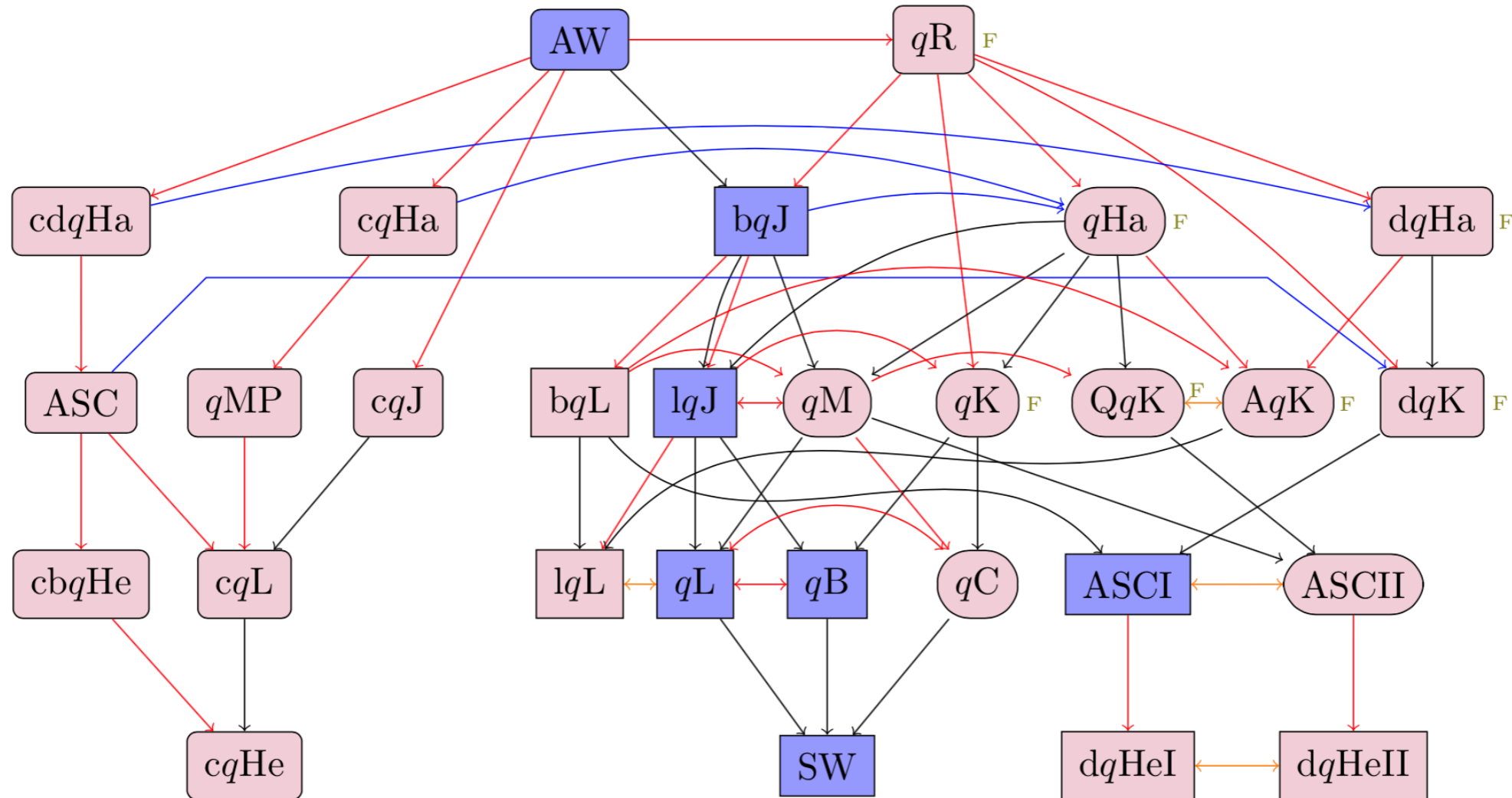
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# Outline

- Present the problem in a wider context ( $q$ -polynomials, or  $q$ -classical polynomials)
- Define the families involved: Al-Salam-Carlitz I, II.
- Explain the basic technique
- Describe the new orthogonality relations
- Present the generating function for such family.

# The Classical Basic Hypergeometric Orthogonal Polynomials



Info: relevant, squared:  $q$ -linear lattice, almost squared: quadratic lattice, oval:  $q^{-1}$ -linear lattice. Red line: particular case, black line: limiting case, orange line: if the first is on  $q$ , the second is on  $q^{-1}$ , blue lines: we discovered (new!), F: blunt orthogonality.

The Al-Salam-Carlitz polynomials  $U_n^{(a)}(x; q)$  were introduced by W. A. Al-Salam and L. Carlitz in [1] as follows:

$$U_n^{(a)}(x; q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^n \frac{(q^{-n}; q)_k (x^{-1}; q)_k q^k x^k}{(q; q)_k a^k}.$$

In fact, these polynomials have a Rodrigues-type formula [4, (3.24.10)]

$$U_n^{(a)}(x; q) = \frac{a^n q^{\binom{n}{2}} (1 - q)^n}{q^n w(x; a; q)} (\mathcal{D}_{q^{-1}})^n [w(x; a; q)], \quad w(x; a; q) := (qx; q)_\infty (qx/a; q)_\infty,$$

where the  $q$ -Pochhammer symbol is defined as

$$(z; q)_0 := 1, \quad (z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k),$$

$$(z; q)_\infty := \prod_{k=0}^{\infty} (1 - zq^k), \quad |z| < 1,$$

and the  $q$ -derivative operator is defined by

$$(\mathcal{D}_q f)(z) := \begin{cases} \frac{f(qz) - f(z)}{(q - 1)z} & \text{if } q \neq 1 \wedge z \neq 0, \\ f'(z) & \text{if } q = 1 \vee z = 0. \end{cases}$$

## Orthogonality Relation

$$\int_a^1 (qx, a^{-1}qx; q)_\infty U_m^{(a)}(x; q) U_n^{(a)}(x; q) d_q x$$

$$= (-a)^n (1-q) (q; q)_n (q, a, a^{-1}q; q)_\infty q^{\binom{n}{2}} \delta_{mn}, \quad a < 0.$$

## Recurrence Relation

$$xU_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a+1)q^n U_n^{(a)}(x; q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x; q).$$

[4] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their  $q$ -analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder.

$$\sum_{k=0}^{\infty} \omega(q^k; a; q) U_m^{(a)}(q^k; q) U_n^{(a)}(q^k; q) q^k \quad a < 0, 0 < q < 1$$

$$- \sum_{k=0}^{\infty} \omega(aq^k; a; q) U_m^{(a)}(aq^k; q) U_n^{(a)}(aq^k; q) aq^k = d_n^2 \delta_{nm}.$$

the  $q$ -Jackson integral [4, (1.15.7)] is defined as

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

where

$$\int_0^a f(x) d_q x := a(1 - q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

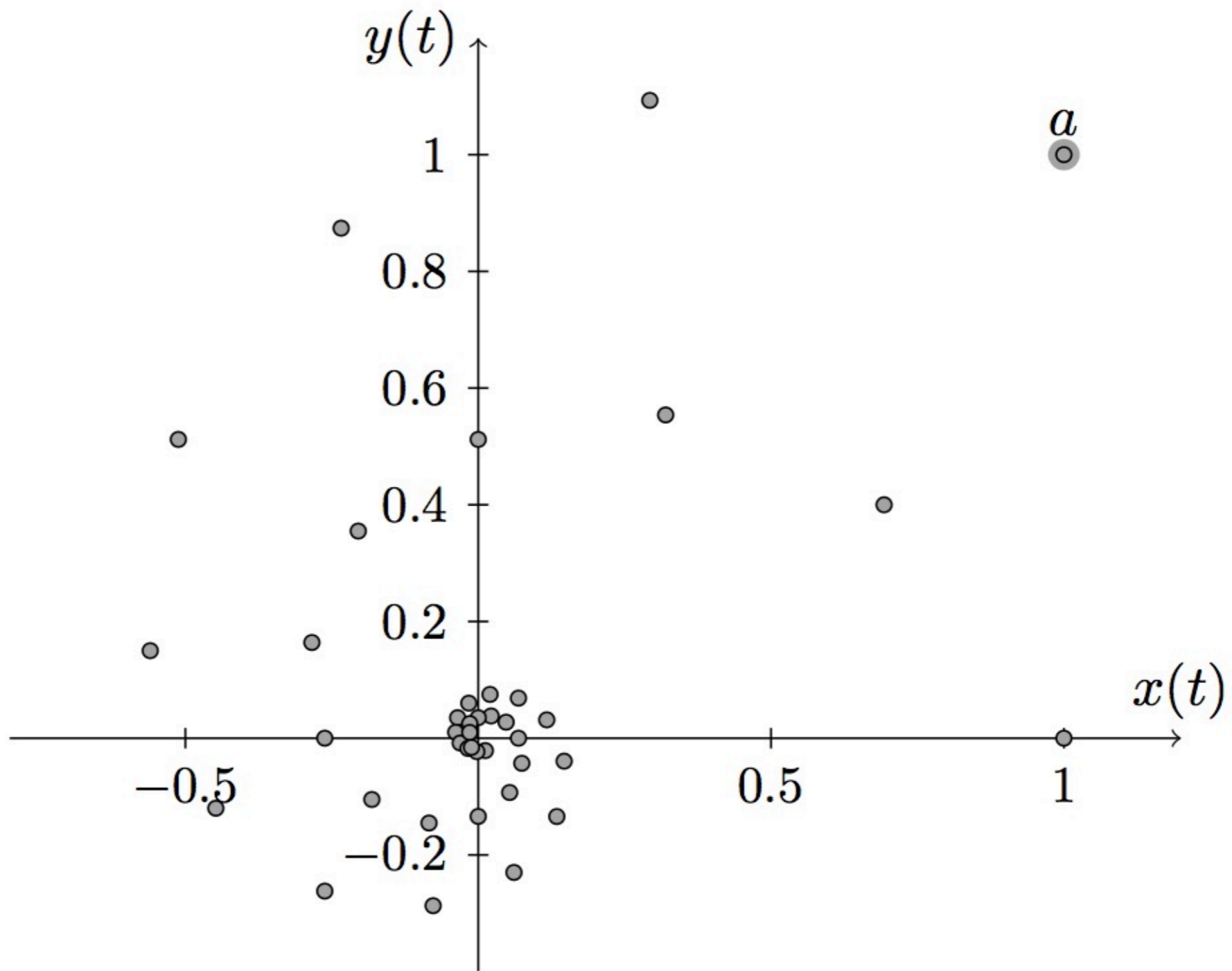


FIGURE 1. Zeros of  $U_{30}^{(1+i)} \left( x; \frac{4}{5} \exp(\pi i/6) \right)$

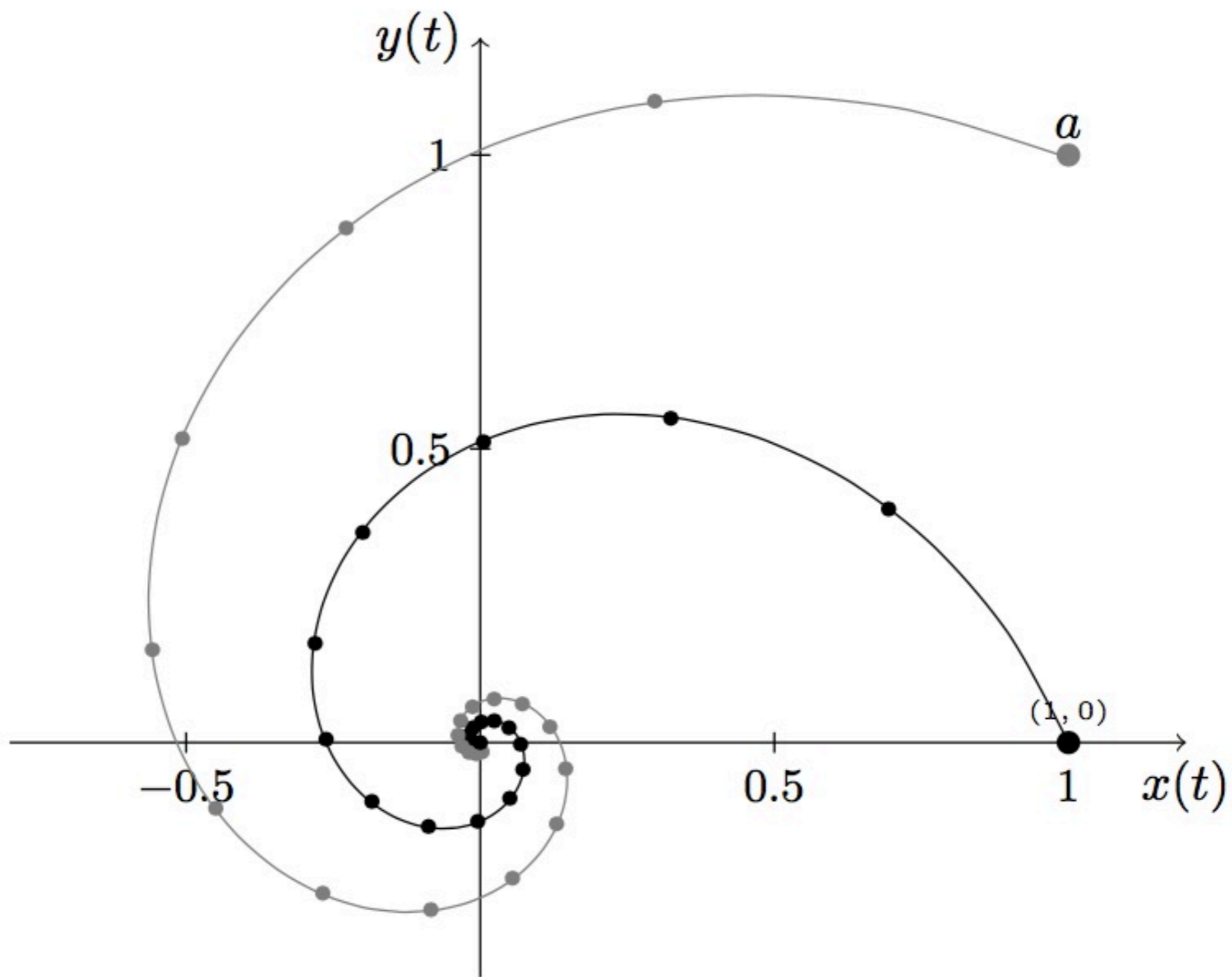
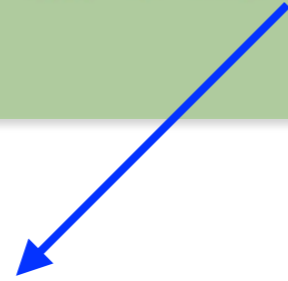


Fig. 2. The lattice  $\{q^k : k \in \mathbb{N}_0\} \cup \{(1 + i)q^k : k \in \mathbb{N}_0\}$  with  $q = 4/5 \exp(\pi i/6)$ .



**Theorem 1.** Let  $a, q \in \mathbb{C}$ ,  $a \neq 0, 1$ ,  $0 < |q| < 1$ , the Al-Salam-Carlitz polynomials are the unique polynomials (up to a multiplicative constant) satisfying the property of orthogonality

$$\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q) w(x; a; q) d_q x = d_n^2 \delta_{n,m}. \quad (2)$$



$$\sum_{k=0}^{\infty} \omega(q^k; a; q) U_m^{(a)}(q^k; q) U_n^{(a)}(q^k; q) q^k \quad a \in \mathbb{C} \setminus \{0, 1\}, \quad |q| < 1$$

$$- \sum_{k=0}^{\infty} \omega(aq^k; a; q) U_m^{(a)}(aq^k; q) U_n^{(a)}(aq^k; q) aq^k = d_n^2 \delta_{nm}.$$

$$\sum_{k=0}^M f(q^k) \mathcal{D}_{q^{-1}} g(q^k) q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1} - 1}$$

$$- \sum_{k=0}^M g(q^{k-1}) \mathcal{D}_{q^{-1}} f(q^k) q^k.$$

From the  $m=n$  case one obtains the following identity:

**Corollary 1.** *Let  $a, q \in \mathbb{C} \setminus \{0\}$ ,  $|q| < 1$ . Then*

$$\sum_{k=0}^{\infty} \left( (q^{k+1}/a; q)_{\infty} - a(aq^{k+1}; q)_{\infty} \right) \frac{q^k}{(q; q)_k} = (a; q)_{\infty} (q/a; q)_{\infty}.$$

## Orthogonality Relation

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{q^{k^2} a^k}{(q; q)_k (aq; q)_k} V_m^{(a)}(q^{-k}; q) V_n^{(a)}(q^{-k}; q) \\ &= \frac{(q; q)_n a^n}{(aq; q)_{\infty} q^{n^2}} \delta_{mn}, \quad 0 < aq < 1. \end{aligned}$$

**Theorem 2.** Let  $a, q \in \mathbb{C}$ ,  $a \neq 0, 1$ ,  $|q| > 1$ . Then, the Al-Salam-Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by

$$\begin{aligned} & \int_a^1 U_n^{(a)}(x; q^{-1}) U_m^{(a)}(x; q^{-1}) (q^{-1}x; q^{-1})_{\infty} (q^{-1}x/a; q^{-1})_{\infty} d_{q^{-1}}x \\ &= (-a)^n (1 - q^{-1}) (q^{-1}; q^{-1})_n (q^{-1}; q^{-1})_{\infty} (a; q^{-1})_{\infty} (q^{-1}/a; q^{-1})_{\infty} q^{-\binom{n}{2}} \delta_{m,n}. \end{aligned}$$

## The remaining cases

**Remark 4.** The  $a = 1$  case is special because it is not considered in the literature. In fact, the linear form associated with the Al-Salam-Carlitz polynomials  $\mathbf{u}$  is quasi-definite and fulfills the Pearson-type distributional equations

$$\mathcal{D}_q[(x-1)^2\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u} \quad \text{and} \quad \mathcal{D}_{q^{-1}}[q^{-1}\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u}.$$

$$xU_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a+1)q^n U_n^{(a)}(x; q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x; q)$$

The  $|q|=1$  case

# **A generalized generating function for Al-Salam-Carlitz polynomials**

$$|q| > 1, \quad p = 1/q$$

**Theorem 3.** Let  $a, b, p \in \mathbb{C} \setminus \{0\}$ ,  $|p| < 1$ ,  $a, b \neq 1$ . Then

$$U_n^{(a)}(x; p) = (-1)^n (p; p)_n p^{-\binom{n}{2}} \sum_{k=0}^n \frac{(-1)^k a^{n-k} (b/a; p)_{n-k} p^{\binom{k}{2}}}{(p; p)_{n-k} (p; p)_k} U_k^{(b)}(x; p).$$

## Generating Functions

$$\frac{(xt; q)_\infty}{(t, at; q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q; q)_n} V_n^{(a)}(x; q) t^n.$$

$$(at; q)_\infty \cdot {}_1\phi_1 \left( \begin{matrix} x \\ at \end{matrix}; q, t \right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(q; q)_n} V_n^{(a)}(x; q) t^n.$$

**Theorem 4.** Let  $a, b, p \in \mathbb{C} \setminus \{0\}$ ,  $|p| < 1$ ,  $a, b \neq 1$ ,  $t \in \mathbb{C}$ ,  $|at| < 1$ . Then

$$(at; p)_\infty {}_1\phi_1 \left( \begin{matrix} x \\ at \end{matrix}; p, t \right) = \sum_{k=0}^{\infty} \frac{p^{k(k-1)}}{(p; p)_k} {}_1\phi_1 \left( \begin{matrix} b/a \\ 0 \end{matrix}; p, atp^k \right) U_k^{(b)}(x; p) t^k, \quad (8)$$

**Theorem 5.** *Let  $a, b, p \in \mathbb{C} \setminus \{0\}$ ,  $t \in \mathbb{C}$ ,  $|at| < 1$ ,  $|p| < 1$ ,  $m \in \mathbb{N}_0$ . Then*

$$\int_a^1 {}_1\phi_1 \left( \begin{matrix} q^{-x} \\ at \end{matrix}; q, t \right) U_m^{(b)}(q^{-x}; p)(q^{-1}x; q^{-1})_\infty (q^{-1}x/a; q^{-1})_\infty dq^{-1}$$

$$= (-bt)^m q^{3\binom{m}{2}} (b; p)_\infty (p/b; p)_\infty {}_1\phi_1 \left( \begin{matrix} b/a \\ 0 \end{matrix}; q, atq^m \right).$$



**Thank you for your attention!**