The orthogonality relations of the Al-Salam-Carlitz polynomials for general parameters

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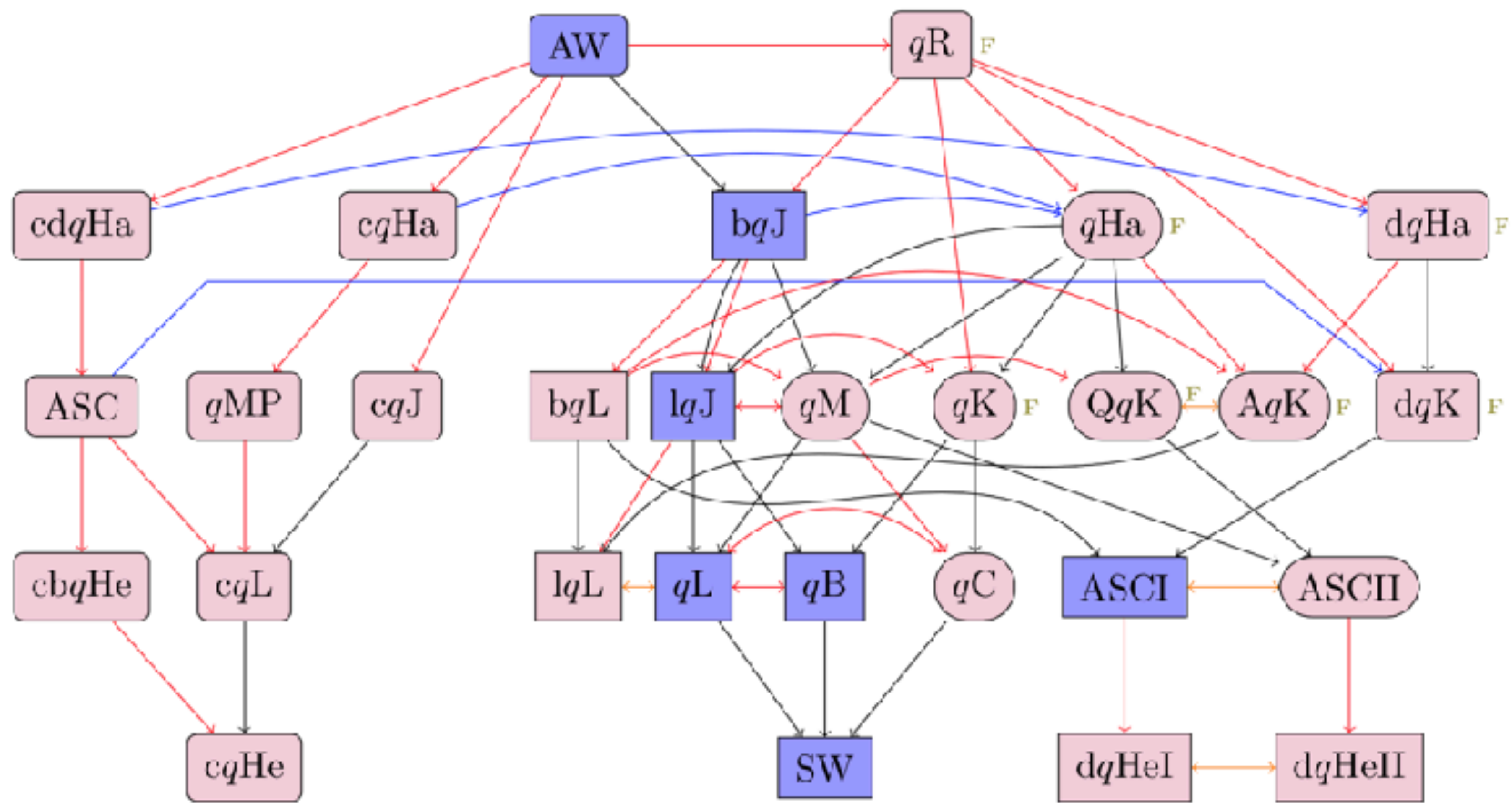
Outline

- Present the problem in a wider context (q-polynomials, or q-classical polynomials)
- Define the families involved: Al-Salam-Carlitz I, II.
- Explain the basic technique
- Describe the new orthogonality relations
- Present the generating function for such family.

The Classical Basic Hypergeometric Orthogonal Polynomials



Info: relevant, squared: $q-$linear lattice, almost squared: quadratic lattice, oval: $q^{-1}-$linear lattice. Red line: particular case, black line: limiting case, orange line: if the first is on q, the second is on q^{-1}, blue lines: we discovered (new!), F: blunt orthogonality.

The Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ were introduced by W. A. Al-Salam and L. Carlitz in [1] as follows:

$$U_n^{(a)}(x;q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} \frac{q^k x^k}{a^k}.$$

In fact, these polynomials have a Rodrigues-type formula [4, (3.24.10)]

$$U_n^{(a)}(x;q) = \frac{a^n q^{\binom{n}{2}} (1-q)^n}{q^n w(x;a;q)} (\mathscr{D}_{q^{-1}})^n [w(x;a;q)], \qquad w(x;a;q) := (qx;q)_\infty (qx/a;q)_\infty,$$

where the q-Pochhammer symbol is defined as

$$(z;q)_0 := 1, \quad (z;q)_n := \prod_{k=0}^{n-1} (1 - zq^k),$$

$$(z;q)_\infty := \prod_{k=0}^{\infty} (1 - zq^k), \quad |z| < 1,$$

and the q-derivative operator is defined by

$$(\mathscr{D}_q f)(z) := \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & \text{if } q \neq 1 \wedge z \neq 0, \\ f'(z) & \text{if } q = 1 \vee z = 0. \end{cases}$$

[1] W. A. Al-Salam and L. Carlitz. Some orthogonal q-polynomials. *Mathematische Nachrichten*, 30:47–61, 1965.

Orthogonality Relation

$$\int_a^1 (qx, a^{-1}qx;q)_\infty U_m^{(a)}(x;q)U_n^{(a)}(x;q)\, d_qx$$
$$= (-a)^n(1-q)(q;q)_n(q,a,a^{-1}q;q)_\infty q^{\binom{n}{2}}\,\delta_{mn}, \quad a<0.$$

Recurrence Relation

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^nU_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q).$$

[4] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder.

$$\sum_{k=0}^{\infty} \omega(q^k;a;q)\,U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)q^k \qquad a<0, 0<q<1$$
$$-\sum_{k=0}^{\infty} \omega(aq^k;a;q)U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)aq^k = d_n^2\delta_{nm}.$$

. the q-Jackson integral [4, (1.15.7)] is defined as

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

where

$$\int_0^a f(x)d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

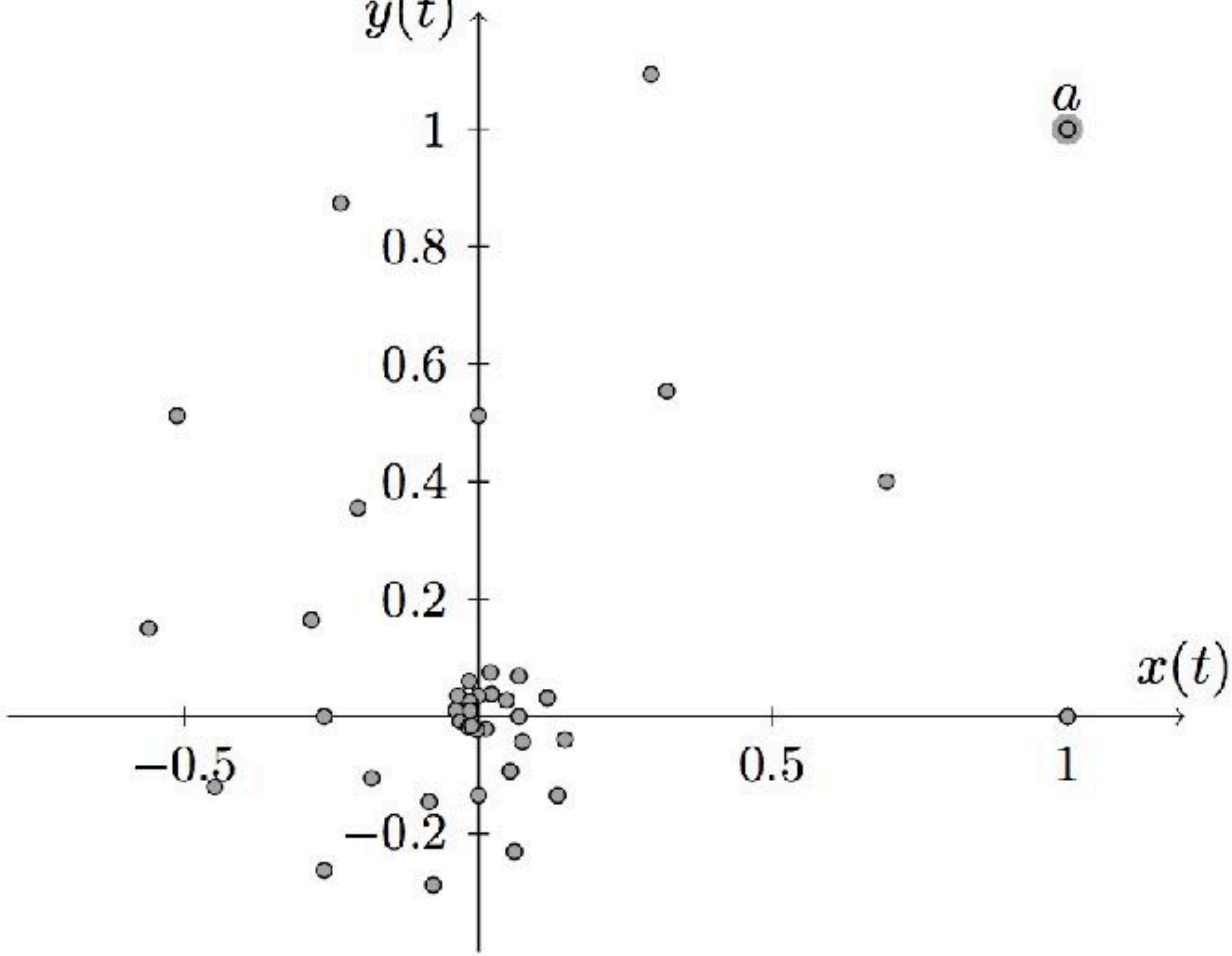


FIGURE 1. Zeros of $U_{30}^{(1+i)}\left(x; \frac{4}{5}\exp(\pi i/6)\right)$

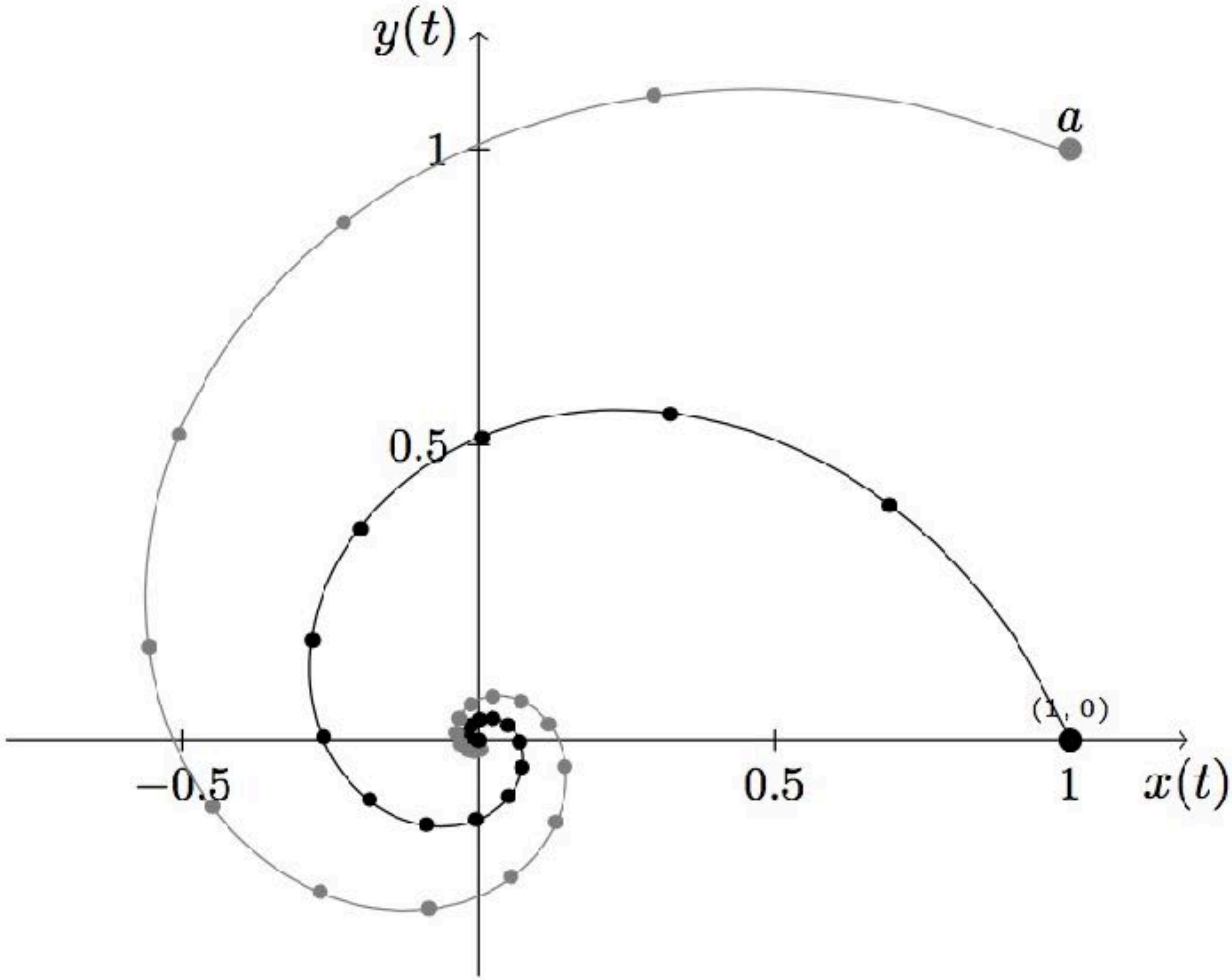


Fig. 2. The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 4/5 \exp(\pi i/6)$.

Theorem 1. *Let $a, q \in \mathbb{C}$, $a \neq 0, 1$, $0 < |q| < 1$, the Al-Salam-Carlitz polynomials are the unique polynomials (up to a multiplicative constant) satisfying the property of orthogonality*

$$\int_a^1 U_n^{(a)}(x;q)U_m^{(a)}(x;q)w(x;a;q)d_qx = d_n^2\delta_{n,m}. \tag{2}$$

$$\sum_{k=0}^{\infty}\omega(q^k;a;q)U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)q^k - \sum_{k=0}^{\infty}\omega(aq^k;a;q)U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q)aq^k = d_n^2\delta_{nm}.$$

$a \in \mathbb{C} \setminus \{0, 1\}, \quad |q| < 1$

$$\sum_{k=0}^{M} f(q^k)\mathscr{D}_{q^{-1}}g(q^k)q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1}-1} - \sum_{k=0}^{M} g(q^{k-1})\mathscr{D}_{q^{-1}}f(q^k)q^k.$$

From the m=n case one obtains the following identity:

Corollary 1. *Let* $a, q \in \mathbb{C} \setminus \{0\}$, $|q| < 1$. *Then*

$$\sum_{k=0}^{\infty} \left((q^{k+1}/a; q)_\infty - a(aq^{k+1}; q)_\infty \right) \frac{q^k}{(q;q)_k} = (a;q)_\infty (q/a;q)_\infty.$$

Orthogonality Relation

$$\sum_{k=0}^{\infty} \frac{q^{k^2} a^k}{(q;q)_k (aq;q)_k} V_m^{(a)}(q^{-k};q) V_n^{(a)}(q^{-k};q)$$
$$= \frac{(q;q)_n a^n}{(aq;q)_\infty q^{n^2}} \delta_{mn}, \quad 0 < aq < 1.$$

Theorem 2. *Let $a, q \in \mathbb{C}$, $a \neq 0, 1$, $|q| > 1$. Then, the Al-Salam-Carlitz polynomials are unique (up to a multiplicative constant) satisfying the property of orthogonality given by*

$$\int_a^1 U_n^{(a)}(x;q^{-1}) U_m^{(a)}(x;q^{-1}) (q^{-1}x;q^{-1})_\infty (q^{-1}x/a;q^{-1})_\infty d_{q^{-1}}x$$
$$= (-a)^n (1-q^{-1})(q^{-1};q^{-1})_n (q^{-1};q^{-1})_\infty (a;q^{-1})_\infty (q^{-1}/a;q^{-1})_\infty \, q^{-\binom{n}{2}} \delta_{m,n}.$$

The remaining cases

Remark 4. The $a = 1$ case is special because it is not considered in the literature. In fact, the linear form associated with the Al-Salam-Carlitz polynomials $\mathbf{u}$ is quasi-definite and fulfills the Pearson-type distributional equations

$$\mathscr{D}_q[(x-1)^2\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u} \quad \text{and} \quad \mathscr{D}_{q^{-1}}[q^{-1}\mathbf{u}] = \frac{x-2}{1-q}\mathbf{u}.$$

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^n U_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q)$$

The $|q|$=1 case

A generalized generating function for Al-Salam-Carlitz polynomials

$|q|>1$, p=1/q

Theorem 3. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$. Then*

$$U_n^{(a)}(x;p) = (-1)^n (p;p)_n p^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k a^{n-k} (b/a;p)_{n-k} p^{\binom{k}{2}}}{(p;p)_{n-k}(p;p)_k} U_k^{(b)}(x;p).$$

Generating Functions

$$\frac{(xt;q)_\infty}{(t,at;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} V_n^{(a)}(x;q)t^n.$$

$$(at;q)_\infty \cdot {}_1\phi_1\left(\begin{matrix} x \\ at \end{matrix}; q, t\right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(q;q)_n} V_n^{(a)}(x;q)t^n.$$

Theorem 4. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $|p| < 1$, $a, b \neq 1$, $t \in \mathbb{C}$, $|at| < 1$. Then*

$$(at;p)_\infty \, {}_1\phi_1\left(\begin{matrix} x \\ at \end{matrix}; p, t\right) = \sum_{k=0}^{\infty} \frac{p^{k(k-1)}}{(p;p)_k} \, {}_1\phi_1\left(\begin{matrix} b/a \\ 0 \end{matrix}; p, atp^k\right) U_k^{(b)}(x;p)t^k, \quad (8)$$

Theorem 5. *Let $a, b, p \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $|at| < 1$, $|p| < 1$, $m \in \mathbb{N}_0$. Then*

$$\int_a^1 {}_1\phi_1\left(\begin{matrix} q^{-x} \\ at \end{matrix}; q, t\right) U_m^{(b)}(q^{-x}; p)(q^{-1}x; q^{-1})_\infty (q^{-1}x/a; q^{-1})_\infty d q^{-1}$$

$$= (-bt)^m q^{3\binom{m}{2}} (b; p)_\infty (p/b; p)_\infty \, {}_1\phi_1\left(\begin{matrix} b/a \\ 0 \end{matrix}; q, atq^m\right).$$

Thank you for your attention!