Number Theory, q-calculus. The Zeta and q-Zeta functions

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0. Preliminars

$\boxed{1}$ Envirorement

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2 Functions

$\Phi_n(x), \quad \zeta(z), \quad [n;\beta]_q \quad \& \quad \zeta_q(z;\beta).$

3 Basic Ideas and Results

- Basic Number Theory functions.
- Basic cyclothomic polynomials identities,
- k-Perfect numbers,
- Factorization of the Zeta function. Identities.
- q-Numbers and q-Zeta functions.

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3 $\pi(x) = \#\{\text{number of primes} \leq \; x\} \backsim \frac{x}{\log x}$,

4 $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt \backsim$ Riemann conjecture.

2. Cyclothomic polynomials

- Let $\eta \in \mathbb{C} : \eta^n = 1$ and $\eta^k \neq 1$ if $k = 0 : n-1$

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▷ **Properties:**

- $\Phi_n(x)$ is irreducible in $\mathbb{Z}[x]$,
- The degree of $\Phi_n(x)$ is $\varphi(n)$,
- $x^n - 1 = \prod_{k|n} \Phi_k(x)$, (*)
- $\Phi_r(x^n) = \prod_{t|n} \Phi_{rt}(x)$, $\gcd(n, r) = 1$,
- $\Lambda(n) = \log(\Phi_n(1))$. (!)

3. Some Applications of $\Phi_n(x)$

- k-perfect numbers. Definition.
  Ex. $k = 2 \to 6$, $k = 3 \to 120$, $k = 4 \to 30240$,
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- Let p, q be two primes s.t. $q | \Phi_n(p)$ then

$$q|n \quad \text{or} \quad \#_q(p) | \gcd(n, q-1).$$

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- **New Function:** Let n be a positive integer,

$$\zeta_\Phi(n) = \prod_{p \text{ prime}} [\Phi_1^{\mu(n)}(1/p)\Phi_n^{-1}(1/p)].$$

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- For p a prime integer $\longrightarrow \zeta_\Phi(p) = \zeta(p)$.
- Some identities using NT functions.

5. q-Calculus and q-NT

- Fixed β, $q \in \mathbb{C}$, $q^\beta \neq 1$, the q^β-number

$$[n;\beta]_q \equiv [n]_{q^\beta} = \frac{q^{n\beta}-1}{q^\beta - 1}.$$

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- $[n;\beta]_q$ is q-**prime** iff exists an irreducible polynomial $f \in \mathbb{Z}[x] : [n;\beta]_q = f(q^\beta)$. And is q-**composite** in other case.

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  And is q-**composite** in other case.
- $\{\overline{[n;\beta]_q} \; : \; n, \, \beta \in \mathbb{Z}\}$ is an UFD.

5. q-Calculus and q-NT (cont.)

- Let's consider a representant of $\overline{[n;\beta]_q}$

$$[\![p_1^{\alpha_1}\cdots p_k^{\alpha_k};\beta]\!]_q := [p_1;\beta]_q^{\alpha_1}\cdots[p_k;\beta]_q^{\alpha_k}.$$

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- In that Field all the q-representations of a functional representation via the q-numbers can be considered "the same", but in the practice: convergence, zeros, applications in approximation theory, ... can present different behavior.

6. Special Functions

1 If we consider a functional equation that verifies a Special Function $f(z)$ we can

- Don't allow the parameter β changes in the q-analog of this equation,

$$\widetilde{\Gamma}_q(z+1) = [z;\beta]_q \widetilde{\Gamma}_q(z),\ \widetilde{\Gamma}_q(0) = 1.$$

- Allow this.

$$\widehat{\Gamma}_q(z+1;\beta) = [z;\beta]_q \widehat{\Gamma}_q(z;z\beta),\ \widehat{\Gamma}_q(0;\beta) = 1.$$

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2 If we have a function like this

$$f(z) = \sum_{n\in\Lambda\subseteq\mathbb{Z}} a(n,z) \rightarrow \overline{f_q(z)} = \sum_{n\in\Lambda\subseteq\mathbb{Z}} a(\overline{[n;\beta]}, z).$$

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$\boxed{3}$ For instance, to recover **Euler Identity** we need

$$[\![n;\beta]\!]_q.$$

7. An application with q-OP

1. Let's consider $h_p(1)$ the q-analog of harmonic series

$$h_p(1) = \sum_{k \geq 1} \frac{1}{p^k - 1}, \quad 1 < p < \infty.$$

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4. After the process he need to calculate the following limit

$$\lim_{n\to\infty} [d_n(p)]^{\frac{1}{n^2}} \text{ where } d_n(p) = \text{lcm}(p-1, \ldots, p^n - 1).$$

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6 $\sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2} n^2 + \mathcal{O}(n \log n)$.

7 $\lim_{n\to\infty} [d_n(p)]^{\frac{1}{n^2}} = p^{3/\pi^2}$.

8 Finally, using this result he obtains an upper bound of the measure of irrationality for this approximation, which was founded by Bundshuh and Väänänen $\rightarrow r(h_p(1)) \leq 1 + \frac{\pi^2+2}{\pi^2-2}$.