Orthogonality of q-polynomials for nonstandard parameters

Roberto S. Costas-Santos
Universidad de Alcalá

Work supported by MCeI grant MTM2009-12740-C03-01

Leganés, 17 March 2011

Outline

1. The Favard's theorem
2. Degenerate version of Favard's theorem
3. An example: The Askey-Wilson Polynomials

The Favard’s theorem

Three-term recurrence relation (TTRR)

- Consider the polynomials $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

with initial conditions $p_{-1}(x) \equiv 0$, $p_0(x) = 1$.

Theorem (Favard) If $\gamma_n \neq 0 \; \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with r_n a non-vanishing normalization factor.

Degenerate version of Favard's theorem

Preliminaries

Let $\mathscr{T}_1 : \mathbb{P}[x] \to \mathbb{P}[x]$ be a linear operator such that

- $\deg \mathscr{T}_1(p) = \deg p - 1$

Preliminaries

Let $\mathscr{T}_1 : \mathbb{P}[x] \to \mathbb{P}[x]$ be a linear operator such that

- $\deg \mathscr{T}_1(p) = \deg p - 1$

- The **monic** polynomials $p_{n,1}$ defined by

$p_{n,1} := \text{const.} \mathscr{T}_1(p_{n+1})$ fulfill the TTRR

$$xp_{n,1}(x) = p_{n+1,1}(x) + \beta_{n,1}p_{n,1}(x) + \gamma_{n,1}p_{n-1,1}(x)$$

so that there exists $\lambda : \{\gamma_{n,1} = 0\} \to \{\gamma_n = 0\}$ strictly increasing with $\lambda(n) > n$.

Preliminaries

Let $\mathscr{T}_1 : \mathbb{P}[x] \to \mathbb{P}[x]$ be a linear operator such that

- $\deg \mathscr{T}_1(p) = \deg p - 1$

- The **monic** polynomials $p_{n,1}$ defined by

$p_{n,1} := \text{const.} \mathscr{T}_1(p_{n+1})$ fulfill the TTRR

$$xp_{n,1}(x) = p_{n+1,1}(x) + \beta_{n,1}p_{n,1}(x) + \gamma_{n,1}p_{n-1,1}(x)$$

so that there exists $\lambda : \{\gamma_{n,1} = 0\} \to \{\gamma_n = 0\}$ strictly increasing with $\lambda(n) > n$.

Consequence: $(p_{n,1})$ is orthogonal with respect to some moments functional $\mathscr{L}_1$.

Iterating process

- $p_{n,k} := \text{const.}\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \text{const.}\mathscr{T}^{(k)}(p_{n+k})$

Iterating process

- $$p_{n,k} := \text{const.}\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \text{const.}\mathscr{T}^{(k)}(p_{n+k})$$
- $$xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k}p_{n,k}(x) + \gamma_{n,k}p_{n-1,k}(x)$$

Iterating process

- $p_{n,k} := \text{const.}\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \text{const.}\mathscr{T}^{(k)}(p_{n+k})$
- $xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k}p_{n,k}(x) + \gamma_{n,k}p_{n-1,k}(x)$
- $\mathscr{L}_k(p_{m,k}p_{n,k}) = 0$ for $n \neq m$

Iterating process

- $p_{n,k} := \mathrm{const.}\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \mathrm{const.}\mathscr{T}^{(k)}(p_{n+k})$
- $xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k}p_{n,k}(x) + \gamma_{n,k}p_{n-1,k}(x)$
- $\mathscr{L}_k(p_{m,k}p_{n,k}) = 0$ for $n \neq m$
- The first n such that $\gamma_{n,k} = 0$ (if it exists) verifies $n < N - k$.

Iterating process

- $p_{n,k} := \text{const}.\mathscr{T}_k(p_{n+1,k-1}) = \cdots = \text{const}.\mathscr{T}^{(k)}(p_{n+k})$
- $xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k}p_{n,k}(x) + \gamma_{n,k}p_{n-1,k}(x)$
- $\mathscr{L}_k(p_{m,k}p_{n,k}) = 0$ for $n \neq m$
- The first n such that $\gamma_{n,k} = 0$ (if it exists) verifies $n < N - k$.

Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Notice $\gamma_{n,N} \neq 0$ for all $n \in \mathbb{N}$.

The proof

Degenerate version of Favard's Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Proof:

The proof

Degenerate version of Favard's Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Proof:

- $\langle p_m, p_n\rangle = 0$ by construction

The proof

Degenerate version of Favard's Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Proof:

- $\langle p_m, p_n\rangle = 0$ by construction

- If $n < N$ then

$$\langle p_n, p_n\rangle = \mathscr{L}_0(p_n p_n) = \gamma_1\gamma_2 \dots \gamma_n \neq 0.$$

The proof

Degenerate version of Favard's Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Proof:

- $\langle p_m, p_n\rangle = 0$ by construction
- If $n < N$ then

$$\langle p_n, p_n\rangle = \mathscr{L}_0(p_n p_n) = \gamma_1\gamma_2 \dots \gamma_n \neq 0.$$

- If $n \geq N$ then

$$\begin{aligned}\langle p_n, p_n\rangle = \ & \mathrm{const.}\mathscr{L}_N(p_{n-N}p_{n-N}) = \\ & \mathrm{const.}\gamma_{n-N,N}\gamma_{n-N-1,N}\dots\gamma_{1,N} \neq 0.\end{aligned}$$

Corollary

Corollary: If $\Lambda = \{n : \gamma_n = 0\}$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \sum_{j\in\mathscr{A}} \mathscr{L}_j(\mathscr{T}^{(j)}(f)\mathscr{T}^{(j)}(g)),$$

being $\mathscr{A} = \{N_0, N_1, \dots\}$ with
$N_{j+1} = N_j + \min\{n : \gamma_{n,N_j} = 0\}$.

The operator $\mathscr{T}$

Among all the possible choices the linear operator $\mathscr{T}$ can be chosen as

- The "Associating operator"

$$\mathscr{T}(p)(x) = \mathscr{L}_0 \left(\frac{p(x) - p(t)}{x - t} \right)$$

($\mathscr{L}_0$ acts on the variable t)

The operator $\mathscr{T}$

Among all the possible choices the linear operator $\mathscr{T}$ can be chosen as

- The "Associating operator"

$$\mathscr{T}(p)(x) = \mathscr{L}_0\left(\frac{p(x) - p(t)}{x - t}\right)$$

($\mathscr{L}_0$ acts on the variable t)

- If (p_n) is classical, then $\mathscr{T}$ is
  - the derivative, or
  - a difference operator.

The example: Askey-Wilson polynomials

Monic Askey-Wilson polynomials

The monic ones are $p_n(x; a, b, c, d; q) \equiv p_n(x)$

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_n(x),$$

with

$$\frac{\gamma_n}{1-q^n} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{4(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$$

Case $abcd \in \{q^{-k} : k \in \mathbb{N}_0\}$ are not considered since they are not normal.

They are symmetric with respect to any rearrangement of the parameters a, b, c, d.

$$\{n \in \mathbb{N} : \gamma_n = 0\} \neq \emptyset \iff ab, ac, \ldots, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$$

$\iff$ they are q-Racah (until now considered as a finite family).

Orthogonality of AW polynomials for $|q| < 1$

$$\int_C p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n\delta_{n,m}$$

where

- W is analytic in $\mathbb{C}$ except at the poles 0,

$$aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0 \quad \text{(the convergent poles)}$$

$$(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0 \quad \text{(the divergent poles)}$$

Orthogonality of AW polynomials for $|q| < 1$

$$\int_C p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n\delta_{n,m}$$

where

- W is analytic in $\mathbb{C}$ except at the poles 0,

$$aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0 \quad \text{(the convergent poles)}$$

$$(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0 \quad \text{(the divergent poles)}$$

- C is the unit circle deformed to separate the convergent form the divergent poles.

The 3 key cases

- Case I: $a^2 = q^{-N+1}$ and

$$b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case II: $ab = q^{-N+1}$ and

$$a^2, b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case III: $ab = q^{-N+1}$, $a^2 = q^{-M}$ with $M \in \{0, 1, \ldots, N-2\}$ and

$$b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.

Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.
Poles:

$$\dots, \pm q^{-M/2-1}, \pm q^{-M/2}, \pm q^{-M/2+1}, \dots, \pm q^{M/2-1}, \pm q^{M/2}, \pm q^{M/2+1}, \dots$$

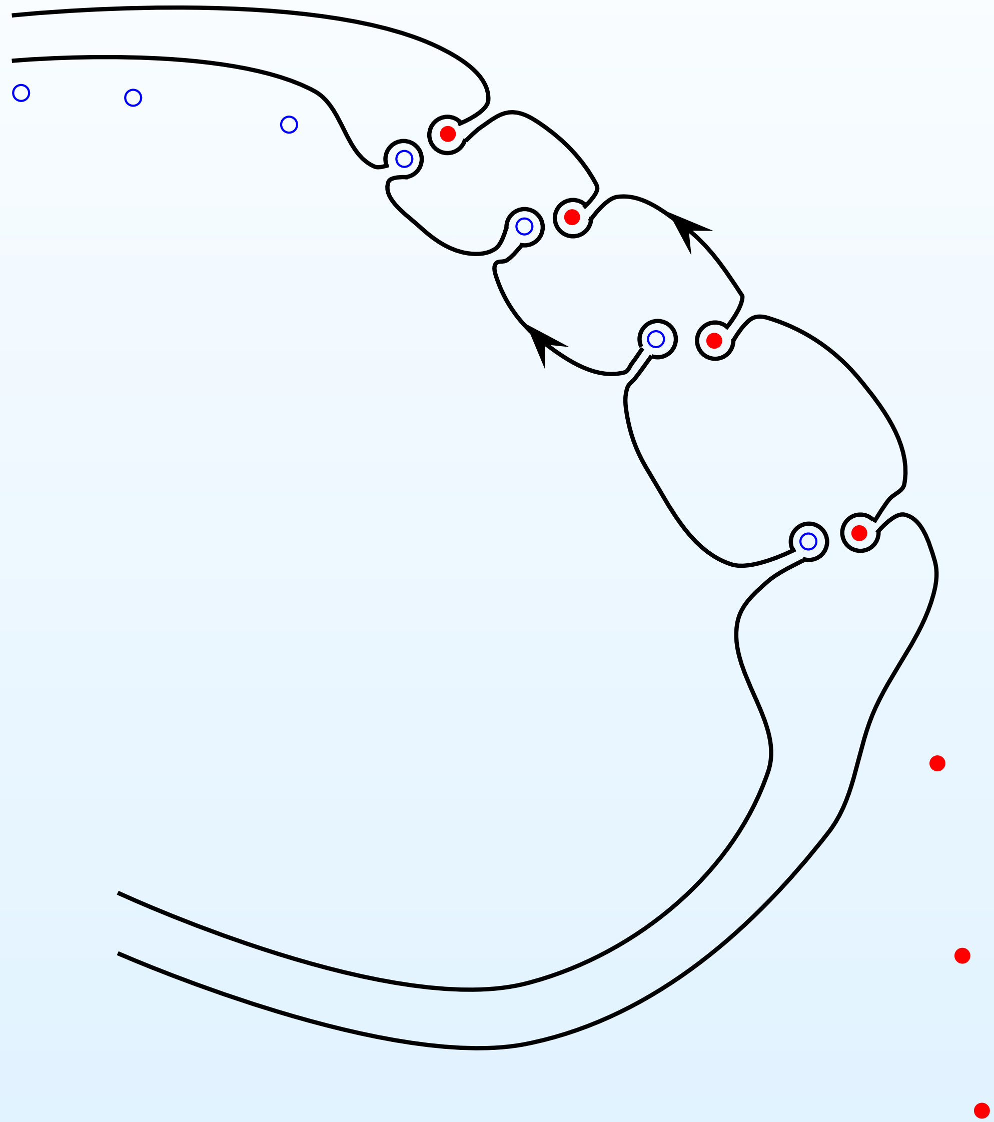
Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.

$$\mathscr{L}_0(p; a, b, c, d) = \lim_{\alpha \to a} \mathscr{L}_0(p; \alpha, b, c, d) = \lim_{\alpha \to a} \int_C p(z)W(z)dz$$

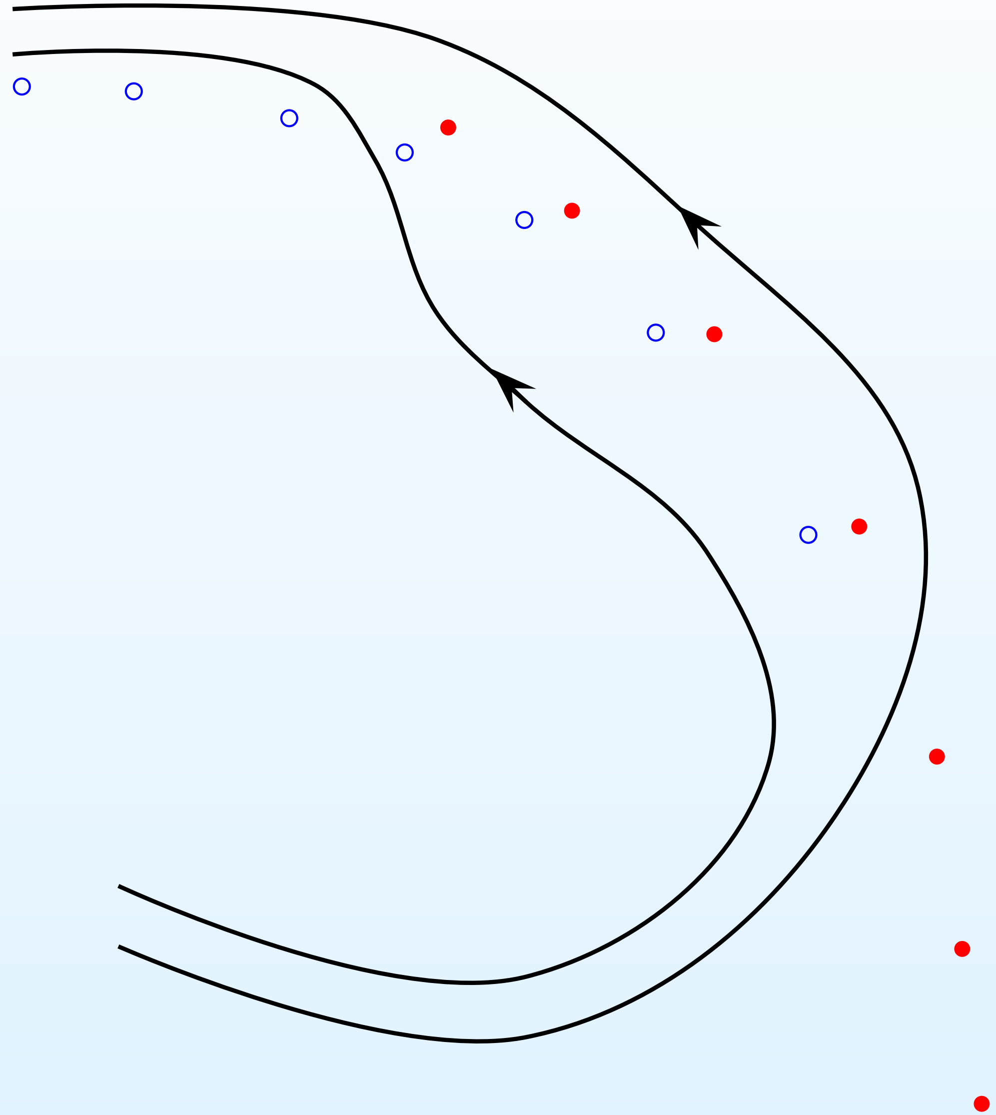
Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.



Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.



Case I: $a^2 = q^{-M}$

Since $\gamma_n \neq 0$ for all n, the orthogonality is given only by $\mathscr{L}_0$.

$$\mathscr{L}_0(p; a, b, c, d) = \lim_{\alpha \to a} \mathscr{L}_0(p; \alpha, b, c, d) = \lim_{\alpha \to a} \int_C p(z)W(z)dz$$

$$\mathscr{L}_0(p; a, b, c, d) = \left(\int_{C_1} + \int_{C_2} \right) p(z)W(z)dz$$

with C_1 and C_2 separating the divergent poles from the convergent ones but the double poles which stand between C_1 and C_2.

Case II: $ab = q^{-N+1}$

In this case $\gamma_N = 0$ (the unique)$\Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

- $\mathscr{L}_0$ is a quadrature rule.

These AW polynomials are the q-Racah polynomials

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1 - a^2q^{2j})}{(cdq^{-N})^j(1 - a^2)} p\left(\frac{q^{-j} + a^2q^{2j}}{2a}\right)$$

Case II: $ab = q^{-N+1}$

In this case $\gamma_N = 0$ (the unique)$\Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

- $\mathscr{L}_0$ is a quadrature rule.

These AW polynomials are the q-Racah polynomials

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1-a^2q^{2j})}{(cdq^{-N})^j(1-a^2)} p\left(\frac{q^{-j}+a^2q^{2j}}{2a}\right)$$

- $\mathscr{T} = \mathscr{D}_q$ the Hahn's operator

$$\mathscr{D}_q(f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \ \wedge \ q \neq 1, \\ f'(z), & z = 0 \ \vee \ q = 1, \end{cases}$$

$$\mathscr{D}^N p_n(x; a, b, c, d; q) = \text{const.} p_{n-N}(x; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q)$$

Case II: $ab = q^{-N+1}$

In this case $\gamma_N = 0$ (the unique)$\Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

- $\mathscr{L}_0$ is a quadrature rule.

  These AW polynomials are the q-Racah polynomials

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1 - a^2q^{2j})}{(cdq^{-N})^j(1 - a^2)} p\left(\frac{q^{-j} + a^2q^{2j}}{2a}\right)$$

- $\mathscr{T} = \mathscr{D}_q$ the Hahn's operator

$$\mathscr{D}_q(f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \wedge q \neq 1, \\ f'(z), & z = 0 \vee q = 1, \end{cases}$$

$$\mathscr{D}^N p_n(x; a, b, c, d; q) = \text{const.} p_{n-N}(x; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q)$$

- $\mathscr{L}_N(p; a, b, c, d) = \mathscr{L}_0(p; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2})$

Case III

$ab = q^{-N+1}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$ with only $\gamma_N = 0 \Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

Case III

$ab = q^{-N+1}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$ with only $\gamma_N = 0 \Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

Orthogonality in this case whole be the same that in case II

$$\widehat{\mathscr{L}_0}(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1-a^2q^{2j})}{(cdq^{-N})^j(1-a^2)} p\left(\frac{q^{-j}+a^2q^{2j}}{2a}\right)$$

but $\widehat{\mathscr{L}_0} \equiv 0$!.

Case III

$ab = q^{-N+1}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$ with only $\gamma_N = 0 \Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

Orthogonality in this case whole be the same that in case II

$$\widehat{\mathscr{L}_0}(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1-a^2q^{2j})}{(cdq^{-N})^j(1-a^2)} p\left(\frac{q^{-j}+a^2q^{2j}}{2a}\right)$$

but $\widehat{\mathscr{L}_0} \equiv 0$!.

The good one:

$$\mathscr{L}_0(p) = \lim_{\alpha \to a} \frac{\widehat{\mathscr{L}_0}(p; \alpha, b, c, d)}{\alpha - a} = \frac{d\widehat{\mathscr{L}_0}(p; \alpha, b, c, d)}{d\alpha}\bigg|_{\alpha=a}.$$

Case III

$ab = q^{-N+1}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$ with only $\gamma_N = 0 \Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

Orthogonality in this case whole be the same that in case II

$$\widehat{\mathscr{L}_0}(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1 - a^2q^{2j})}{(cdq^{-N})^j(1 - a^2)} p\left(\frac{q^{-j} + a^2q^{2j}}{2a}\right)$$

but $\widehat{\mathscr{L}_0} \equiv 0$!.

The good one:

$$\mathscr{L}_0(p) = \lim_{\alpha \to a} \frac{\widehat{\mathscr{L}_0}(p; \alpha, b, c, d)}{\alpha - a} = \frac{d\widehat{\mathscr{L}_0}(p; \alpha, b, c, d)}{d\alpha}\Big|_{\alpha = a}.$$

The result is a quadrature rule with simple and double nodes

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

- $q = \exp(2M\pi/NI)$. In this case $\gamma_{jN} = 0, j \in \mathbb{N}$.

  - Spiridonov and Zhedanov found $\mathscr{L}_0$

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

- $q = \exp(2M\pi/NI)$. In this case $\gamma_{jN} = 0, j \in \mathbb{N}$.
  - Spiridonov and Zhedanov found $\mathscr{L}_0$
  - For $n > N$

$$\mathscr{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q)$$

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

- $q = \exp(2M\pi/NI)$. In this case $\gamma_{jN} = 0, j \in \mathbb{N}$.

    - Spiridonov and Zhedanov found $\mathscr{L}_0$
    - For $n > N$

$$\mathscr{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q)$$

- $\mathscr{L}_j(p(\centerdot)) = \mathscr{L}_0(p((-1)^M \centerdot))$

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

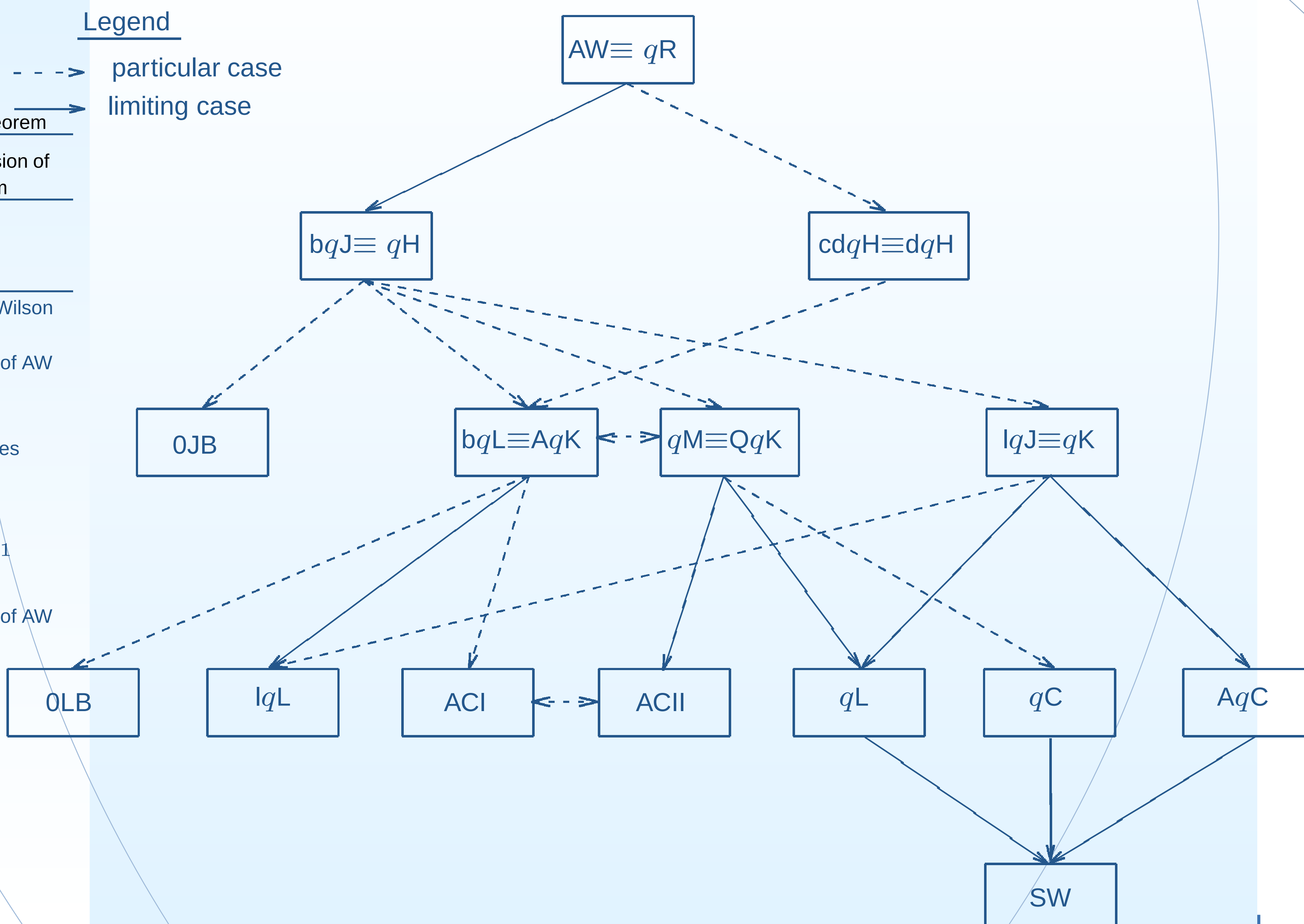
- $q = \exp(2M\pi/NI)$. In this case $\gamma_{jN} = 0, j \in \mathbb{N}$.
  - Spiridonov and Zhedanov found $\mathscr{L}_0$
  - For $n > N$

$$\mathscr{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q)$$

  - $\mathscr{L}_j(p(\cdot)) = \mathscr{L}_0(p((-1)^M \cdot))$

- For the rest of the values of q the result keeps UNKNOWN.

The scheme


Legend
particular case
limiting case
AW≡ qR
bqJ≡ qH
cdqH≡dqH
0JB
bqL≡AqK
qM≡QqK
lqJ≡qK
0LB
lqL
ACI
ACII
qL
qC
AqC
SW