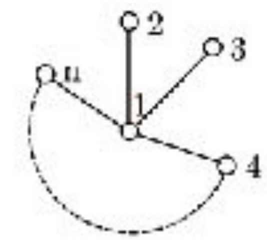


# MATRICES TOTALLY POSITIVE RELATIVE TO A TREE


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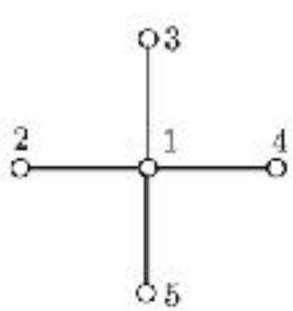
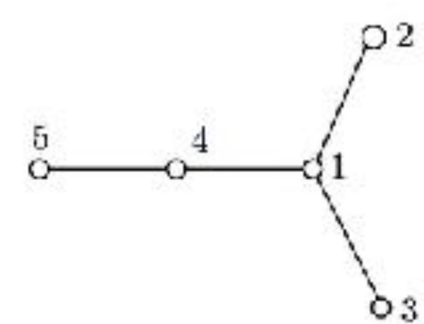



**Joint work with Charles R. Johnson**

Leganés, December 15th 2016

# Introduction

- **DEF:** A matrix is called totally positive (**TP**) if every minor of it is positive.
- We will be interested in submatrices of a given matrix that are TP, or permutation similar to TP.
- Thus, we will be interested in permuted submatrices, identified by ordered index lists.
- Suppose that $T$ is a labelled tree on $n$ vertices and A is an $n$-by-$n$ matrix.
- **DEF:** If **P** is an induced path of $T$ , by **A[P]** we mean A[α] in which α consists of the indices of the vertices of P in the order in which they appear along P .
- Since everything we discuss is independent of reversal of order, there in no ambiguity.
- **DEF:** For a given labelled tree $T$ on $n$ vertices, we say that **A** is **T-TP** if, for every path P in $T$, A[P] is TP.

# Introduction

- For a T-TP matrix, properly less is required than for a TP matrix.

- Also, like TP matrices, T-TP matrices are entry-wise positive. This follows because there exists a path connecting vertices *i* and *j* in tree *T*, so that every entry in the corresponding T-TP matrix is in a submatrix that is, by definition, TP.

- Since all the entries in a TP matrix are positive, then A has positive coefficients.

# TP matrices properties

- Among them is the fact that the eigenvalues are real, positive and distinct.

- The largest one is the Perron root and its eigenvector may be taken to be positive.

- The fact that this property of a TP matrix holds for T-TP matrices is clear from the fact that the entries are positive.

- The eigenvectors of the remaining eigenvalues alternate in sign subject to well-defined requirements, and, in particular, the eigenvector, associated with the smallest eigenvalue, alternates in sign as: $(+,-,+,-,...,)$. This is because the inverse, or adjoint, has a checkerboard sign pattern and the Perron root of the alternating sign signature similarity of the inverse is the inverse of the smallest eigenvalue of the original TP matrix.

# T-TP matrices (sign pattern)

- If *T* is a Path, the sign pattern of the eigenvector concerning the *smallest* eigenvalue may be viewed as alternation associated with each edge of *T*, i.e. if {i, j} is an edge of *T* , then $v_i v_j < 0$ for the eigenvector v associated with the smallest eigenvalue.

- The vector v is signed according to the labelled tree *T* on *n* vertices if, whenever {i, j} is an edge of *T* , then $v_i v_j < 0$.
  This means that v is totally nonzero and that the sign pattern of v is uniquely determined, up to a factor of −1.

- We know that the eigenvector associated with the smallest eigenvalue of a TP matrix is signed according to the standardly labelled path *T* (relative to which the TP matrix is T-TP).

- Neumaier originally conjectured that all eigenvectors should be signed as those of a TP matrix, and J. Garloff relayed to us that for any tree *T* , the eigenvector associated with the smallest eigenvalue of a T-TP matrix should be signed according to the labelled tree T. We have proved that the conjecture is false.

## PART I

(With Boris Tadchiev)

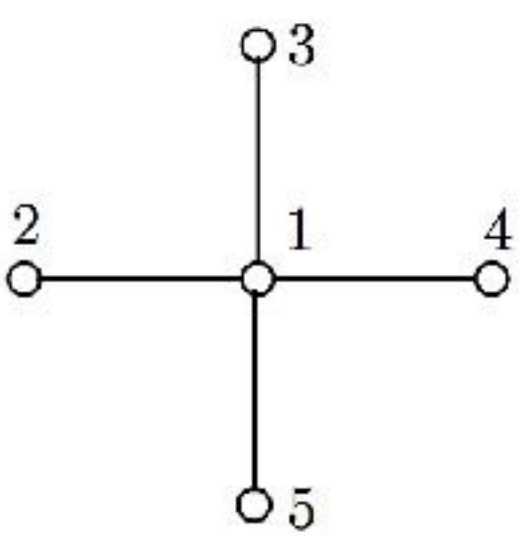
THE **STARS** and the **POTCHFORK**

# The stars on 4 vertices

- **Theorem**. For any labelled tree T on fewer than 5 vertices, any T-TP matrix has smallest eigenvalue that is real and a totally nonzero eigenvector that is signed according to T.

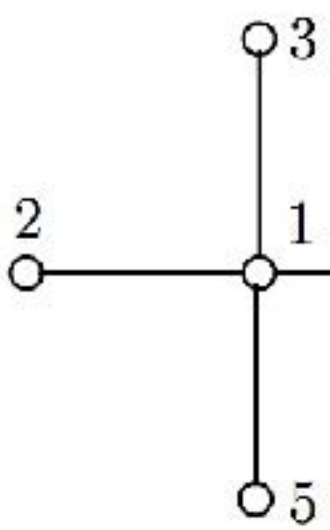
**Sketch of the proof**

1. Then we wish to show that if A is T-TP, then the sign pattern of $\tilde{A}$ is




$$\begin{bmatrix} + & - & - & - \\ - & + & + & + \\ - & + & + & + \\ - & + & + & + \end{bmatrix}.$$

$$
\begin{aligned}
\tilde{a}_{3,2} &= (-1)^{3+2} \det A[1,3,4;1,2,4] \\
&= -\det A[3,1,4;2,1,4] \\
&= -\frac{\det A[3,1;2,1] \det A[1,4;1,4] - \det A[3,1;1,4] \det A[1,4;2,1]}{\det A[1;1]}.
\end{aligned}
$$




$$
\tilde{a}_{3,2} = -\frac{(-)(+) - (+)(+)}{(+)} > 0.
$$

$$
\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & - & - & - \\ - & + & + & + \\ - & + & + & + \\ - & + & + & + \end{bmatrix}.
$$

Taking into account (1.1), we get that

$$
\begin{bmatrix} \tilde{a}_{11} & - & - & - \\ - & + & + & + \\ - & + & + & + \\ - & + & + & + \end{bmatrix}
\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}
=
\begin{bmatrix} \det A & 0 & 0 & 0 \\ 0 & \det A & 0 & 0 \\ 0 & 0 & \det A & 0 \\ 0 & 0 & 0 & \det A \end{bmatrix}
$$

and multiplying the first row of $\tilde{A}$ by the second column of $A$ we get

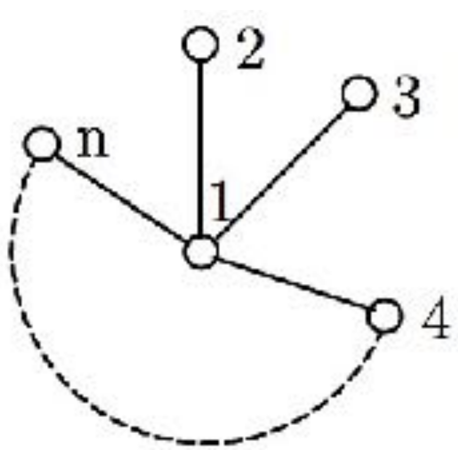
$$
\tilde{a}_{11} + (-) + (-) + (-) = 0,
$$

# The stars on n vertices

- **Theorem**. Let *T* be a star on *n* vertices. Suppose that *A* is T-TP and that all the submatrices of *A* associated with the deletion of **pendant** vertices are P-matrices. Then, the smallest eigenvalue of *A* is real, has multiplicity one and has an eigenvector signed according to *T*.

## Sketch of the proof

1. Then we wish to show that if A is T-TP, then the sign pattern of $\tilde{A}$ is




$$\tilde{A} = \left[\begin{array}{c|cccc} + & - & \cdots & - \\ \hline - & + & \cdots & + \\ \vdots & \vdots & \ddots & \vdots \\ - & + & \cdots & + \end{array}\right].$$

$$\begin{aligned}\tilde{a}_{ij} &= (-1)^{i+j}(-1)^{j-1}(-1)^{i-2}\det A[i,1,\aleph;j,1,\aleph]\\ &= (-1)^{2(i+j-1)-1}\det A[i,1,\aleph;j,1,\aleph] > 0.\end{aligned}$$

$\det A[i,1,\aleph;j,1,\aleph] =$

$$\frac{\det A[i,1,\aleph';j,1,\aleph']\det A[1,\aleph;1,\aleph] - \det A[i,1,\aleph';1,\aleph]\det A[1,\aleph;j,1,\aleph']}{\det A[1,\aleph';1,\aleph']}.$$

$$\tilde{A} = \left[\begin{array}{c|ccc} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \hline \tilde{a}_{21} & + & \cdots & + \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & + & \cdots & + \end{array}\right].$$

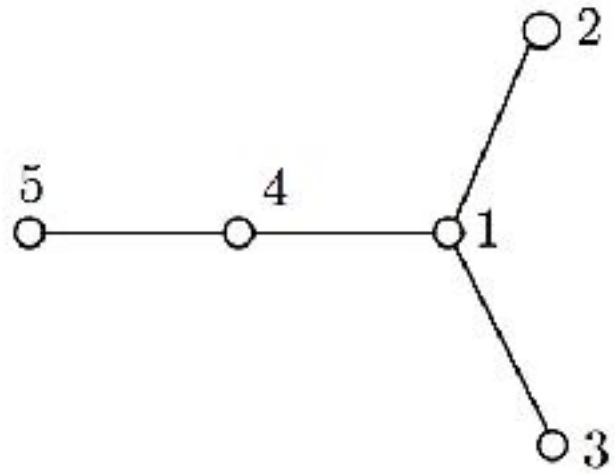
Taking into account (1.1), we get that

$$\begin{bmatrix} \tilde{a}_{11} & - & - & - \\ - & + & + & + \\ - & + & + & + \\ - & + & + & + \end{bmatrix}\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 & 0 \\ 0 & \det A & 0 & 0 \\ 0 & 0 & \det A & 0 \\ 0 & 0 & 0 & \det A \end{bmatrix}$$

and multiplying the first row of $\tilde{A}$ by the second column of $A$ we get

$$\tilde{a}_{11} + (-) + (-) + (-) = 0,$$

# Bad example: The **Pitchfork**




$$A = \begin{bmatrix} 55 & 77 & 10 & 17 & 49 \\ 40 & 84 & 3 & 1 & 8 \\ 57 & 74 & 86 & 15 & 47 \\ 94 & 2 & 8 & 86 & 58 \\ 48 & 41 & 4 & 4 & 78 \end{bmatrix}$$

$$\lambda_5 \approx -6.16.$$

$$\mathbf{x} \approx \begin{bmatrix} -2.98 \\ 1.21 \\ -0.02 \\ 2.39 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 88 & 50 & 35 & 78 & 38 \\ 50 & 48 & 19 & 27 & 11 \\ 35 & 19 & 41 & 13 & 6 \\ 78 & 27 & 13 & 86 & 44 \\ 38 & 11 & 6 & 44 & 59 \end{bmatrix}$$

$$\lambda_5 \approx -2.54,$$

$$\tilde{A} = \begin{bmatrix} 42023084 & -27857784 & -2494736 & -6756454 & -17014640 \\ -18274672 & 7046528 & 1241168 & 2950496 & 7815680 \\ 2070092 & 1908264 & -5017752 & 386110 & 1240248 \\ -35907780 & 21866360 & 2481608 & 951670 & 18111768 \\ -14519176 & 12220096 & 1012872 & 2538312 & 279496 \end{bmatrix}.$$

$$\mathbf{x} \approx \begin{bmatrix} -68.08 \\ 32.75 \\ 26.69 \\ 45.57 \\ 1 \end{bmatrix}.$$

# PART II

Pendant vertices
and
positive determinant

# New definitions

- **DEF:** For a given labelled tree *T* on n vertices, we say that A is **pendent-P** relative to *T* if all principal submatrices, associated with the deletion of pendent vertices, one at a time, are P-matrices.

- **DEF:** For a given labelled tree *T* on n vertices, we say that A is ***T*-positive** if it is T-TP and pendent-P relative to *T*.

# THE GENERAL RESULT

- Let $S_\sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with σ signed according to *T* .
- **Theorem.** Let *T* be a labelled tree on *n* vertices and A be *T*-positive with det A > 0. Then

  $$\text{sign}(\det A(i;j)) = (-1)^{i+j}\sigma_i\sigma_j$$

  in which σ is signed according to T.
- **Corollary** If *T* is a tree on n vertices and A is *T*-positive with detA > 0. Then $S_\sigma A^{-1} S_\sigma$ is entry-wise positive. Therefore, A satisfies the Neumaier conclusion.

# Scketch of the proof

**Lemma 7.** *Given a matrix $A \in M_n(\mathbb{R})$, then for any three distinct integers $i$, $j$, $k$, with $1 \leq i, j, k \leq n$, we have*

$$\widetilde{a}_{k,i} \det A[i, N_{i,j,k}; i, N_{i,j,k}] + \widetilde{a}_{k,j} \det A[j, N_{i,j,k}; i, N_{i,j,k}] + \widetilde{a}_{k,k} \det A[k, N_{i,j,k}; i, N_{i,j,k}] = 0.$$

$$\operatorname{sign}(\det A(i;j)) = (-1)^{i+j}\sigma_i\sigma_j \iff \operatorname{sign}(\widetilde{a}_{i,j}) = \sigma_i\sigma_j$$

For the last lemma we need to use *Jacobi's identity* [3, (0.8.4.1)]

$$\det A[\alpha;\beta] = (-1)^{p(\alpha,\beta)} \det A \;\; \det A^{-1}[N \setminus \beta; N \setminus \alpha],$$

where $|\alpha| = |\beta|$, and $p(\alpha,\beta) = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$.

$$\det \widetilde{A}[j,p;i,p] = \begin{vmatrix} \widetilde{a}_{j,i} & \widetilde{a}_{j,p} \\ \widetilde{a}_{p,i} & \widetilde{a}_{p,p} \end{vmatrix} = \widetilde{a}_{j,i}\widetilde{a}_{p,p} - \widetilde{a}_{j,p}\widetilde{a}_{p,i},$$

# Where can I find the results?

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 211-221, April 2009

ELA
http://math.technion.ac.il/iic/ela

MATRICES TOTALLY POSITIVE RELATIVE TO A TREE*

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Matrices totally positive relative to a tree, II

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# Open problems

# Thank you and Merry Christmas