Seminario Dpto. Matemáticas – Universidad Carlos III de Madrid

Exceptional Orthogonal Polynomials from the Semi-classical point of view

Roberto Santiago Costas Santos

Universidad de Alcalá

Joint work with Jessica Stewart Kelly (Christopher Newport University, VA)

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A brief history

- It caught my attention
- I want to know a bit about it

Connecting the dots

- The Type III exceptional Laguerre polynomials

The surprise ... the present

A brief history

about **LUCK**

First, it caught my attention

1. V EIBPOA in Mexico, 2015 (David Gómez-Ullate),
2. 14th OPSFA in Kent, 2017 (David Gómez-Ullate, Jessica Stewart),
3. In 2018, I went to College of William and Mary and ... and then, I started understanding

- Gómez-Ullate, David; Grandati, Yves; Milson, Robert. Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials. *J. Phys. A* **47** (2014), no. 1, 015203, 27 pp. MR3146977
- García-Ferrero, Ma. Ángeles; Gómez-Ullate, David. Oscillation theorems for the Wronskian of an arbitrary sequence of eigenfunctions of Schrödinger's equation. *Lett. Math. Phys.* **105** (2015), no. 4, 551–573. MR3323666
- Liaw, Constanze; Littlejohn, Lance L.; Milson, Robert; Stewart, Jessica. The spectral analysis of three families of exceptional Laguerre polynomials. *J. Approx. Theory* **202** (2016), 5–41. MR3440071
- and almost every reference connected with these ones.... Ryu Sasaki, Satoru Odake, M.M. Crum, V.É. Adler, etc, etc

Connecting the dots

The Example

We know that the Laguerre polynomials are orthogonal with respect to the weight:

$$w^{\mathtt{L}}(x;\alpha) = x^{\alpha}\exp(-x),$$

which satisfies the Pearson equation

$$\frac{d}{dx}\left(x\, w^{\mathtt{L}}(x;\alpha)\right) = (\alpha + 1 - x)\, w^{\mathtt{L}}(x;\alpha).$$

This polynomials are eigenfunctions of the operator

$$\mathscr{L}^{\alpha} = x\frac{d^2}{dx^2} + (\alpha + 1 - x)\frac{d}{dx}.$$

with eigenvalues

$$\lambda_n^{\mathtt{L}} = -n.$$

$$\mathscr{L}^{\alpha}[L_n^{\alpha}(x)] = x\,(L_n^{\alpha}(x))'' + (\alpha+1-x)\,(L_n^{\alpha}(x))' = -nL_n^{\alpha}(x).$$

In the construction we realised we need polynomials $\widetilde{y}_n(x;\alpha)$ eigenfunctions of

$$\widetilde{\mathscr{L}}^{\alpha} = x\frac{d^2}{dx^2} - (\alpha+1-x)\frac{d}{dx}.$$

$$x\left(L_n^{-\alpha-2}(-x)\right)'' - (\alpha+1-x)\left(L_n^{-\alpha-2}(-x)\right)' = -nL_n^{-\alpha-2}(-x),$$

therefore

$$\widetilde{y}_n(x;\alpha) = L_n^{-\alpha-2}(-x).$$

These polynomials also fulfill the differential equation:

$$x\left(L_n^{-\alpha-1}(-x)\right)' - (\alpha+1-x)L_n^{-\alpha-1}(-x) = (n+1)L_n^{-\alpha-2}(-x).$$

We consider the operators:

$$\begin{aligned}\mathscr{A}^{III,\alpha} &:= xL_m^{-\alpha}(-x)\frac{d}{dx} - (m+1)L_m^{-\alpha-1}(-x),\\ \mathscr{B}^{III,\alpha} &:= \frac{1}{L_m^{-\alpha}(-x)}\frac{d}{dx}.\end{aligned}$$

Remember, the operators we have considered are

$$\mathscr{A}^{III,\alpha} := x L_m^{-\alpha}(-x)\frac{d}{dx} - (m+1)L_m^{-\alpha-1}(-x),$$
$$\mathscr{B}^{III,\alpha} := \frac{1}{L_m^{-\alpha}(-x)}\frac{d}{dx}.$$

They fulfill the following identities:

$$\mathscr{B}^{III,\alpha} \circ \mathscr{A}^{III,\alpha} = \mathscr{L}^{\alpha} - (m+1)\mathscr{I}.$$

Moreover, we have

$$\begin{aligned}\mathscr{A}^{III,\alpha} \circ \mathscr{B}^{III,\alpha} &= x\frac{d^2}{dx^2} + \left((\alpha+1-x-x') - 2x\frac{\widetilde{y}_m'(x;\alpha)}{\widetilde{y}_m(x;\alpha)}\right)\frac{d}{dx} \\ &= \mathscr{L}_m^{III,\alpha-1}.\end{aligned}$$

We define the (Exceptional) eigenfunctions

$$
\begin{aligned}
\mathscr{A}^{III,\alpha} &= xL_m^{-\alpha}(-x)\frac{d}{dx} - (m+1)L_m^{-\alpha-1}(-x) \\
&= \frac{x\left(L_m^{-\alpha}(-x)\right)^2}{\omega^{\mathtt{L}}(x;\alpha)}\frac{d}{dx}\frac{\omega^{\mathtt{L}}(x;\alpha)}{L_m^{-\alpha}(-x)} \\
&= \frac{x\left(\widetilde{y}_m(x;\alpha+1)\right)^2}{\omega^{\mathtt{L}}(x;\alpha)}\frac{d}{dx}\frac{\omega^{\mathtt{L}}(x;\alpha)}{\widetilde{y}_m(x;\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
L_{m,n}^{III,\alpha}(x) &= \mathscr{A}^{III,\alpha+1}[-L_{n-m-1}^{\alpha+1}(x)] \\
&= xL_{n-m-2}^{\alpha+2}(x)L_m^{-\alpha-1}(-x) + (m+1)L_{n-m-1}^{\alpha+1}(x)L_{m+1}^{-\alpha-2}(-x).
\end{aligned}
$$

The type I exceptional Laguerre pol. seq. of codimension m

$$L_{m,n}^{III,\alpha}(x) = xL_{n-m-2}^{\alpha+2}(x)L_m^{-\alpha-1}(-x) + (m+1)L_{n-m-1}^{\alpha+1}(x)L_{m+1}^{-\alpha-2}(-x).$$

$$\mathscr{B}^{III,\alpha}[L_{m,n}^{III,\alpha}(x)] = \frac{\left(L_{m,n}^{III,\alpha}(x)\right)'}{L_m^{-\alpha-1}(-x)} = nL_{n-m-1}^{\alpha+1}(x),$$

therefore

$$L_{m,n}^{III,\alpha}(x) = L_{m,n}^{III,\alpha}(0)+n\int_0^x L_{n-m-1}^{\alpha+1}(\xi)L_m^{-\alpha-1}(-\xi)\,d\xi\,, \quad n \in \mathbb{N}_0\backslash\Lambda,$$

where Λ is the exclusionary set with order m.

The surprise ... the present

The common pattern

$$x\left(L_m^\alpha(-x)\right)' + \alpha L_m^\alpha(-x) = (\alpha+m)L_m^{\alpha+1}(-x).$$

$$\begin{aligned}\mathscr{A}^{I,\alpha} &= L_m^\alpha(-x)\frac{d}{dx} - L_m^{\alpha+1}(-x) = e^x\frac{d}{dx}e^{-x}L_m^\alpha(-x)\\ &= \frac{1\left(L_m^\alpha(-x)\right)^2}{e^{-x}}\frac{d}{dx}\frac{e^{-x}}{L_m^\alpha(-x)},\end{aligned}$$

$$\mathscr{B}^{I,\alpha} = \frac{1}{L_m^\alpha(-x)}\left(x\frac{d}{dx}+(1+\alpha)\right) = \frac{1}{L_m^\alpha(-x)x^\alpha}\frac{d}{dx}x x^\alpha.$$

and

$$\begin{aligned}L_{m,n}^{II,\alpha}(x) &= xL_m^{-\alpha-1}(x)L_{n-m-1}^{\alpha+2}(x) + (m-\alpha-1)L_m^{-\alpha-2}(x)L_{n-m}^{\alpha+1}(x)\\ &= \mathscr{A}^{II,\alpha-1}[L_{n-m}^{\alpha-1}(-x)].\end{aligned}$$

$$x\left(L_m^{\alpha}(-x)\right)' + \alpha L_m^{\alpha}(-x) = (\alpha+m)L_m^{\alpha+1}(-x).$$

$$\begin{aligned}
\mathscr{A}^{II,\alpha} &= xL_m^{-\alpha}(x)\frac{d}{dx} + (\alpha-m)L_m^{-\alpha-1}(x) \\
&= \frac{x\left(L_m^{-\alpha}(x)\right)^2}{x^{\alpha}}\frac{d}{dx}\frac{x^{\alpha}}{L_m^{-\alpha}(x)}, \\
\mathscr{B}^{II,\alpha} &= \frac{1}{L_m^{-\alpha}(x)}\left(\frac{d}{dx}-1\right) = \frac{1}{L_m^{-\alpha}(x)e^{-x}}\frac{d}{dx}1e^{-x}.
\end{aligned}$$

$$\frac{d}{dx}\left((1-x^2)\omega^{\mathsf{J}}(x;\alpha,\beta)\right) = \left(\alpha-\beta-x(\alpha+\beta-2)\right)\omega^{\mathsf{J}}(x;\alpha,\beta),$$

where

$$\omega^{\mathsf{J}}(x;\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}.$$

$$\begin{aligned}
\mathscr{A}^{I,\alpha,\beta} &= (1-x^2)P_m^{-\alpha,-\beta}(x)\frac{d}{dx} + 2(m+1)P_{m+1}^{-\alpha-1,-\beta-1}(x) \\
&= \frac{(1-x^2)\left(P_m^{-\alpha,-\beta}(x)\right)^2}{\omega^{\mathsf{J}}(x;\alpha,\beta)}\frac{d}{dx}\frac{\omega^{\mathsf{J}}(x;\alpha,\beta)}{P_m^{-\alpha,-\beta}(x)}, \\
\mathscr{B}^{I,\alpha,\beta} &= \frac{1}{P_m^{-\alpha,-\beta}(x)}\frac{d}{dx}.
\end{aligned}$$

$$(1-x)\frac{d}{dx}P_n^{\alpha,\beta}(x)+(\alpha+n)P_n^{\alpha-1,\beta+1}(x)=\alpha P_n^{\alpha,\beta}(x).$$

$$\begin{aligned}
\mathscr{A}^{II,\alpha,\beta} &= (1-x)P_m^{-\alpha,\beta}(x)\frac{d}{dx}+(m-\alpha)P_m^{-\alpha-1,\beta+1}(x)\\
&= \frac{(1-x)\left(P_m^{-\alpha,\beta}(x)\right)^2}{(1-x)^\alpha}\frac{d}{dx}\frac{(1-x)^\alpha}{P_m^{-\alpha,\beta}(x)},\\
\mathscr{B}^{II,\alpha,\beta} &= \frac{1}{P_m^{-\alpha,\beta}(x)}\left((1+x)\frac{d}{dx}+(1+\beta)\right)\\
&= \frac{1}{(1+x)^\beta P_m^{-\alpha,\beta}(x)}\frac{d}{dx}(1+x)(1+x)^\beta.
\end{aligned}$$

The exceptional operators have a common (nice) pattern

$$\mathscr{A} = \frac{p_1(x)\,(\phi_m(x))^2}{\omega_1(x)}\frac{d}{dx}\frac{\omega_1(x)}{\phi_m(x)}$$
$$\mathscr{B} = \frac{1}{\phi_m(x)\omega_2(x)}\frac{d}{dx}p_2(x)\omega_2(x).$$

if $p(x) := p_1(x)p_2(x)$, then these operators fulfill the identities:

$$\mathscr{B}\circ\mathscr{A} = p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + \mu_m,$$
$$\mathscr{A}\circ\mathscr{B} = p(x)\frac{d^2}{dx^2} + \left(q(x) - p'(x) - 2p\frac{(\phi_m(x))'}{\phi_m(x)}\right)\frac{d}{dx}.$$

$$\mathscr{A} = \frac{p_1(x)\left(\phi_m(x)\right)^2}{\omega_1(x)} \frac{d}{dx} \frac{\omega_1(x)}{\phi_m(x)}$$

$$\mathscr{B} = \frac{1}{\phi_m(x)\omega_2(x)} \frac{d}{dx} p_2(x)\omega_2(x).$$

$$\widetilde{\mathscr{A}} = \frac{\left(\phi_m(x)\right)^2}{\omega_1(x)} \frac{d}{dx} \frac{p_1(x)\omega_1(x)}{\phi_m(x)}$$

$$\widetilde{\mathscr{B}} = \frac{p_2(x)}{\phi_m(x)\omega_2(x)} \frac{d}{dx} \omega_2(x).$$

$$\widetilde{\mathscr{A}} \circ \widetilde{\mathscr{B}} = p(x)\frac{d^2}{dx^2} + \left(q(x) - 2p\frac{\left(\phi_m(x)\right)'}{\phi_m(x)}\right)\frac{d}{dx}.$$

$$\widetilde{\mathscr{A}} = \frac{(\phi_m(x))^2}{\omega_1(x)} \frac{d}{dx} \frac{p_1(x)\omega_1(x)}{\phi_m(x)}$$

$$\widetilde{\mathscr{B}} = \frac{p_2(x)}{\phi_m(x)\omega_2(x)} \frac{d}{dx} \omega_2(x).$$

Since $\widetilde{\mathscr{A}}[p_{n-f(m)}(x)] = p_{n,m}(x)$, and $\widetilde{\mathscr{B}}[p_{n,m}(x)] = p_{n-f(m)}(x)$, then we obtain integrating by parts, and by using the operators $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$

$$\int_\Gamma p_{n,m}(x) p_{k,m}(x) \frac{\omega_2(x)}{\omega_1(x)\,(\phi_m(x))^2}\, dx = -\int_\Gamma p_n(x) p_k(x) \frac{\omega_2(x)}{\omega_1(x)}\, dx.$$

$$L_{m,n}^{I,\alpha}(x) = (\alpha+n)\,x^{-\alpha}\int_0^x \xi^{\alpha-1}L_{n-m}^{\alpha-1}(\xi)L_m^{\alpha-1}(-\xi)\,d\xi$$

$$= L_m^{\alpha}(-x)L_{n-m}^{\alpha-1}(x) + L_m^{\alpha-1}(-x)L_{n-m-1}^{\alpha}(x).$$

$$L_{m,n}^{II,\alpha}(x) = e^x L_{m,n}^{II,\alpha}(0) + A_n\,e^x\int_0^x e^{-\xi}L_{n-m}^{\alpha+1}(\xi)L_m^{-\alpha-1}(\xi)\,d\xi$$

$$= xL_m^{-\alpha-1}(x)L_{n-m-1}^{\alpha+2}(x) + (m-\alpha-1)L_m^{-\alpha-2}(x)L_{n-m}^{\alpha+1}(x).$$

where $A_n = (\alpha+n-2m+1)$.

You can get similar expressions for all the classical families.

$$\sum_{n=m}^{\infty} L_{m,n}^{I,\alpha}(x)t^n = \left\{tL_m^{\alpha-1}(-x) + (1-t)L_m^{\alpha}(-x)\right\} t^m(1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right).$$

$$\sum_{n=m}^{\infty} L_{m,n}^{II,\alpha}(x)t^n = \left\{xtL_m^{-\alpha-1}(-x) + (1-t)(m-\alpha-1)L_m^{-\alpha-2}(-x)\right\} \times t^m(1-t)^{-\alpha-3} \exp\left(\frac{xt}{t-1}\right).$$

$$\sum_{n=m+1}^{\infty} L_{m,n}^{III,\alpha}(x)t^n = \left\{xtL_m^{-\alpha-1}(-x) + (1-t)(m+1)L_{m+1}^{-\alpha-2}(-x)\right\} \times t^{m+1}(1-t)^{-\alpha-3} \exp\left(\frac{xt}{t-1}\right)$$

$$\phi_m(x)p_{n,m}(x) = \sum_{\nu=-m-1}^{m+1} a_{\nu,n,m}p_{n+\nu,m}(x), \quad n \in \mathbb{N}_0 \setminus \Lambda.$$

Thank you for your attention