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The connection between the Riemann Zeta function and the Legendre polynomials

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- Outline

Classical orthogonal polynomials

Outline

1. Classical orthogonal polynomials and their q-analogues
2. The Riemann Zeta function. The approximation
3. Numerical calculations

Classical orthogonal polynomials

The basics

- Let (P_n) be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1,\ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

- COP: Jacobi, Hermite, Laguerre, Bessel.
  Discrete-COP: Hahn, Racah, Meixner, Krawtchouk, Charlier, Askey-Wilson, q-Racah, q-Hahn, Al Salam Carlitz I, II, etc.

The padé approximants

Goal:

To compute the “diagonal” pade approximant of r functions

$$f_j(z) \sim \sum_{k=0}^{\infty} c_{k,j} \frac{1}{z^{k+1}}, \quad j = 1, \ldots, r, \quad z_0 = \infty$$

1. type I: To compute $A_{n_1}, \ldots, A_{n_r}$, where $\deg A_{n_j} \leq n_j - 1$ and a polynomial $B_{\vec{n}}$ so that

$$A_{n_1}(z) f_1(z) + \cdots + A_{n_r}(z) f_r(z) - B_{\vec{n}}(z) = O\left(\frac{1}{z^{|\vec{n}|}}\right).$$

2. type II: To compute a polynomial $\deg P_{\vec{n}} \leq |n|$, and polynomials $Q_{\vec{n},1}, \ldots, Q_{\vec{n},r}$ so that:

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = O\left(\frac{1}{z^{n_j+1}}\right), \quad j = 1, 2, \ldots, r.$$