SOBOLEV ORTHOGONAL POLYNOMIALS: CONNECTION FORMULAS AND ZEROS

17th Orthogonal Polynomials, Special Functions and Applications

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Goal of this talk

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2. We are going to consider the Sobolev-type inner product

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where $\mathbf{u}$ is a classical linear form, $\nu_1, \ldots, \nu_M \in \mathbb{N}$, and $\mu_1, \ldots, \mu_M, c_1, \ldots, c_M \in \mathbb{R}$.

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3. A main aim is to present some connection formulas related to orthogonal Sobolev-type polynomials as general as possible.

4. Show some numerical experiments with the zeros of Sobolev-type Krwatchouk polynomials.

WHAT IS ...

a connection formulae

Given two sequences of polynomials (p_n) and (q_n), the connection formula is a way to "connect" such polynomial sequences:

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$$q_n = \sum_{k=0}^{n} \lambda_{n,k} p_k, \qquad n = 0, 1, 2, \ldots.$$

One can obtain the connection coefficients in several ways. For example, if $(p_n) =$ops$(\mathbf{u})$ then for $n \in \mathbb{N}_0$, one has

$$\lambda_{n,k} = \frac{\langle \mathbf{u}, q_n p_k \rangle}{\langle \mathbf{u}, p_k p_k \rangle}, \qquad k = 0, 1, \ldots, n.$$

Let $(p_n) =$ops$(\mathbf{u})$, and let us consider the Sobolev-type inner product[1]

$$(P, Q) = \langle \mathbf{u}, PQ \rangle + \mathbb{P}(c)^t A \mathbb{Q}(c),$$

where A is a positive semidefinite matrix, $c \in \mathbb{R}$, and for P, $\mathbb{P}^t(x)$ denotes the row matrix $(P(x), P'(x), \ldots, P^{(r)}(x))$.

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The following connection formula holds:

$$Q_n(x) = P_n(x) - P_n^t(c)(I + A\mathbb{H}_{n-1})^{-1} A \mathbb{K}_{n-1}(x, c).$$

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$$(f, g) = \langle \mathbf{u}, fg \rangle + \lambda f(c) g(c).$$

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$$Q_n(x) = P_n(x) - \frac{\lambda P_n(c)}{1+\lambda K_{n-1}(c,c)} K_{n-1}(x,c),$$

where

$$K_m(x,y) = \sum_{k=0}^{m} \frac{1}{\langle \mathbf{u}, p_k^2\rangle} p_k(x)p_k(y).$$

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What can connection formula …

`be useful for?`

What can the connection formulas be useful for?

1. Hypergeometric representation.
2. Zeros.
3. Asymptotic behavior.
4. Others.

Let us consider the big q-Jacobi case.

We consider the inner product

$$(f,g) = \langle \mathbf{u}^{\mathrm{bqJ}}, fg\rangle + \lambda f(cq)g(cq).$$

The Krall big q-Jacobi polynomials admit the following representation[3]

$$P_n^{\lambda}(x;a,b,c;q) = D_n(x)\,{}_5\phi_4\left(\begin{matrix} q^{-n}\,,\ abq^{n+1}\,,\ q^{1-\alpha_1}\,,\ q^{1-\alpha_2}\,,\ x \\ aq^2\,,\ cq^2\,,\ q^{-\alpha_1}\,,\ q^{-\alpha_2} \end{matrix}\ \middle|\ q;q\right).$$

[3] This can be found in [3, §3.1]

The are several types of `connection formulae`

We consider a classical linear form $\mathbf{u}$ such that

$$\mathcal{D}(\phi\mathbf{u}) = \psi\mathbf{u},$$

where $\mathcal{D}$ is lowering operator and $\deg\phi \leq 2$, $\deg\psi = 1$.

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Next, we consider the Sobolev-type inner product[4]

$$\langle f, g\rangle_S = \langle \mathbf{u}, fg\rangle + \sum_{j=1}^{M} \mu_j f^{(\nu_j)}(c_j) g^{(\nu_j)}(c_j), \qquad f, g \in \mathbb{P}. \quad (1)$$

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We consider $(p_n) =$ops($\mathbf{u}$), $(Q_n) =$ops($\langle \bullet, \bullet \rangle_S$).

[4] The next connection formulae can be found in [4]

First, we can express the inner product (1) as

$$\langle f, g\rangle_S = \langle \mathbf{u}, fg\rangle + (\mathbb{D}f)^t D\mathbb{D}g, \tag{2}$$

where $\mathbb{D}$ is the vector differential operator defined as

$$\mathbb{D}f := \left(f^{(\nu_1)}(x)\Big|_{x=c_1}, f^{(\nu_2)}(x)\Big|_{x=c_2}, \ldots, f^{(\nu_M)}(x)\Big|_{x=c_M} \right)^t,$$

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and D is the diagonal matrix with entries $\mu_1, \ldots, \mu_M$.

The following connection formula holds:

$$\boxed{Q_n(x) = P_n(x) - \mathbb{P}_n^t(\mathbb{I} + D\mathbb{K}_{n-1})^{-1} D\mathbb{K}_{n-1}(x)}$$

where $\mathbb{K}_m(x) = \mathbb{D}_y K_m(x, y)$, and $\mathbb{K}_m = \mathbb{D}^t \mathbb{K}_m(x)$.

Lemma

*Let $M \in \mathbb{N}$, **u** be a classical linear form.*

Let $c_1, c_2, \ldots, c_M \in \mathbb{R}$, $\nu_1, \nu_2, \ldots, \nu_M \in \mathbb{N}_0$, and

let us denote by $(Q_n(x))$ the sequence of orthogonal polynomials with respect to the inner product (1).

If c_i is not a zero of $Q_n(x)$, for $i = 1, 2, \ldots, M$ and for all $n \in \mathbb{N}_0$ then, there exists a polynomial, namely $\zeta(x)$, such that

$$\mathbb{D}\left(\zeta(x) S_n^{\vec{\mu}}(x; \vec{\nu}, \vec{c})\right) = \vec{0}$$

holds.

[5] A similar result can be found in [1, Lemma 2.1]

A REMARK ABOUT THE POLYNOMIAL $\zeta(x)$

Observe that if all the c_i's are all different then

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Without loss of generality, we denote by $\zeta(x)$ to the polynomial of minimum degree, namely $\deg \zeta = \nu$, among all nonzero polynomials satisfying the conditions of the former Lemma.

A little bit more about $\zeta(x)$

It is clear that

$$\langle \zeta(x) f(x), g(x) \rangle_S = \langle \mathbf{u}, \zeta(x) f(x) g(x) \rangle = \langle f(x), \zeta(x) g(x) \rangle_S.$$

It is clear that

$$\langle \zeta(x)f(x), g(x)\rangle_S = \langle \mathbf{u}, \zeta(x)f(x)g(x)\rangle = \langle f(x), \zeta(x)g(x)\rangle_S.$$

Let $(\zeta_j(x))_{j=0}^{\nu}$ be a sequence of polynomials such that

- $\deg \zeta_k(x) = k$,
- $\zeta_k(x) | \zeta_{k+1}(x)$ for $k = 0, 1, \ldots, \nu - 1$,
- $\zeta_\nu(x) = \zeta(x)$.

Second type of connection formulae[6]

Let $\left(P_n^{[\zeta_j^2]}(x)\right) =ops(\zeta_j^2\mathbf{u})$, then

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Let $\left(P_n^{[\zeta_j^2]}(x)\right) =ops(\zeta_j^2\mathbf{u})$, then

If the following conditions hold

$$P_n(c_j)P_{n-1}^{[\zeta_1^2]}(c_j)\cdots P_{n-\nu}^{[\zeta_\nu^2]}(c_j) \neq 0, \quad j = 1, 2, ..., M, \tag{3}$$

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then, there exists a family of coefficients $(\lambda_{j,n})_{j=0}^{\nu}$, not all identically zero, such that, for any $n\geq\nu$, we have

$$\boxed{Q_n(x)=\sum_{j=0}^{\nu}\lambda_{j,n}\,\zeta_j(x)P_{n-j}^{[\zeta_j^2]}(x)}$$

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There exists a family of coefficients $(\xi_{j,n})_{j=0}^{\nu}$, not all identically zero, such that, for any $n \geq \nu$, we have

$$\boxed{Q_n(x) = \sum_{j=0}^{\nu} \xi_{j,n} P_n^{(j)}(x)}$$

Numerical experiment:

the Sobolev-type

Krawtchouck case

A numerical experiment about zeros

Let us consider the inner product:

$$(f, g) = \langle \mathbf{u}^{\mathsf{K}}, fg \rangle + \lambda \Delta^i f(c) \Delta^i g(c) + \mu \Delta^j f(d) \Delta^j g(d).$$

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In such a case we have the following limit polynomials:

$$r^{\mu}_{n,i,j}(x;p,N) = \lim_{\lambda\to\infty} K_{n;i,j}(x;p,N)$$

$$= K_n(x;p,N) - \frac{\begin{vmatrix} \Delta^i K_n(c;p,N) & \mu K^{(i,j)}_{n-1}(c,d) \\ \Delta^j K_n(d;p,N) & 1+\mu K^{(j,j)}_{n-1}(d,d) \end{vmatrix}}{\begin{vmatrix} K^{(i,i)}_{n-1}(c,c) & \mu K^{(i,j)}_{n-1}(c,d) \\ K^{(j,i)}_{n-1}(d,c) & 1+\mu K^{(j,j)}_{n-1}(d,d) \end{vmatrix}} K^{(0,i)}_{n-1}(x,c)$$

$$- \mu \frac{\begin{vmatrix} K^{(i,i)}_{n-1}(c,c) & \Delta^i K_n(c;p,N) \\ K^{(j,i)}_{n-1}(d,c) & \Delta^j K_n(d;p,N) \end{vmatrix}}{\begin{vmatrix} K^{(i,i)}_{n-1}(c,c) & \mu K^{(i,j)}_{n-1}(c,d) \\ K^{(j,i)}_{n-1}(d,c) & 1+\mu K^{(j,j)}_{n-1}(d,d) \end{vmatrix}} K^{(0,j)}_{n-1}(x,d),$$

Another technical result

Lemma

The following identities hold:

- *If* $\lambda, \mu \neq 0$, *then*

$$(1+\eta_n+\nu_n)K_{n;i,j}(x;p,N) = K_n(x;p,N) + \eta_n r^{\mu}_{n,i,j}(x;p,N) + \nu_n S_{n;i,j}(x;p,N),$$

$$(1+\xi_n+\zeta_n)K_{n;i,j}(x;p,N) = K_n(x;p,N) + \xi_n R^{\lambda}_{n,i,j}(x;p,N) + \zeta_n S_{n;i,j}(x;p,N),$$

where $\eta_n, \nu_n, \xi_n, \zeta_n \geq 0$ *for all n.*

- *If* $\lambda \neq 0$ *and* $\mu = 0$, *then*

$$(1+\eta_n)K_{n;i,j}(x;p,N) = K_n(x;p,N) + \eta_n r_{n;i}(x;p,N),$$

where

$$\eta_n = \frac{\Delta^i K_n(c;p,N)}{1 + K^{(i,i)}_{n-1}(c,c)}, \quad n = 1, 2, \ldots \tag{4}$$

Some calculations

The collection of zeros of Krawtchouck for $p = 1/2$ and $N = 12$ is

We are going to run Wolfram Mathematica with the code

```
N[SolveValues[KravchSob[x,7,p,12,A,B,i,j,c,d]==0,x],5]
```

1. For a few cases we set $p = \frac{1}{2}$, $A = B = 10^{55}$, $c = -3$, $d = 15$:

$$i = j = 1, \quad \{-3.9243, 1.7234, 3.9228, 6.0561, 8.1881, 10.378, 17.094\}$$

$$i = j = 5, \quad \{-25.694, 1.5457, 3.8067, 6.0257, 8.2422, 10.494, 55.080\}$$

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2. Now we consider $i \neq j$:

   $i = 3,\ j = 2, \ \{-7.6243, 1.5838, 3.8178, 6.0013, 8.1841, 10.416, 20.244\}$

   $i = 2,\ j = 3, \ \{-5.4426, 1.7307, 3.9854, 6.1711, 8.3441, 10.549, 24.914\}$

But ....

1. We set $p = \frac{1}{2}$, $c = -3$, $d = 15$:

$$i = j = 1, A = 0.1, B = 1 \quad \{0.841, 2.79, 4.76, 6.75, 8.76, 10.8, 375\}$$

$$i = j = 2, A = 1, B = 0.1 \quad \{-66.1, 1.19, 3.24, 5.26, 7.25, 9.22, 11.2\}$$

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   $i = j = 2, A = 1, B = 0.1 \quad \{-66.1, 1.19, 3.24, 5.26, 7.25, 9.22, 11.2\}$

2. Now we consider $i \neq j$, $c = -3$, $d = 2.6$:

   $i = 3,\ j = 4,\ A = 1,\ B = 0.1, \{-11.3, 1.18, 3.22, 5.23, 7.22, 9.19, 11.1\}$

   $i = 4,\ j = 3,\ A = 0.1,\ B = 1, \{-0.766, 2.43, 4.76, 6.99, 9.13, 11.1, 43.8\}$

3. We set $p = \frac{1}{2}$, $c = -7$, $A = B = 10^{55}$, $d = 2.6$:

   $i = j = 5,\ \{-27.035, -2.5249, 1.9076, 4.1716, 6.3594, 8.5240, 10.697\}$

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THANK YOU for your attention!