MULTI-INTEGRAL REPRESENTATIONS FOR JACOBI FUNCTIONS OF THE FIRST AND SECOND KIND

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Part I

The Jacobi polynomials and functions

1. The Jacobi polynomials, $P_n^{(\alpha,\beta)}(z)$ are eigenfunctions of

$$H := (1 - z^2)\frac{d^2 w}{dz^2} + (\beta - \alpha - z(\alpha + \beta + 2))\,\frac{dw}{dz},$$

where α, $\beta \in \mathbb{C}$, and with associated eigenvalues

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4. $$\frac{dw}{dz}P_n^{(\alpha,\beta)}(z) \quad \propto \quad P_{n-1}^{(\alpha+1,\beta+1)}(z).$$

The Jacobi functions (1st kind)

1. The Jacobi polynomials can be written by using hypergeometric series as

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right).$$

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2. The Jacobi functions are a natural extension of the Jacobi polynomials (single Gauss hypergeometric series)

$$P_\gamma^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)} \, {}_2F_1\left(\begin{matrix} -\gamma, \gamma+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2} \right).$$

where $z \in \mathbb{C} \setminus (-\infty, -1]$.

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3. One can obtain different representations for the Jacobi functions (Pfaff's, Euler's transformations).

The Jacobi functions (1st kind)

1. One of such expressions is

$$P_{\gamma}^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)} \left(\frac{2}{z+1}\right)^{\alpha+\beta+\gamma+1} {}_2F_1\left(\begin{matrix}\alpha+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix}; \frac{z-1}{z+1}\right)$$

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2. Note in order to calculate certain integrals we need

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$$P_\gamma^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}$$

   where $\alpha+\gamma \notin -\mathbb{N}$.

3. Moreover, the Jacobi functions are eigenfunctions of

$$H := (1-z^2)\frac{d^2w}{dz^2} + (\beta-\alpha-z(\alpha+\beta+2))\frac{dw}{dz},$$

   where α, $\beta \in \mathbb{C}$, and with associated eigenvalues

$$\lambda_\gamma = -\gamma(\gamma+\alpha+\beta+1), \quad \gamma \in \mathbb{C}.$$

The Jacobi functions (2nd kind)

1. The Jacobi functions of the second kind have the following single Gauss hypergeometric series representation:

$$Q_{\gamma}^{(\alpha,\beta)}(z) = \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+2\gamma+2)(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}\,{}_2F_1\left(\begin{matrix}\gamma+1, \alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right),$$

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where $z \in \mathbb{C} \setminus [-1,1]$.

2. Integral representation:

$$Q_\gamma^{(\alpha,\beta)}(z) = \frac{1}{2^{\gamma+1}(z-1)^\alpha(z+1)^\beta}\int_{-1}^{1}\frac{(1-t)^{\alpha+\gamma}(1+t)^{\beta+\gamma}}{(z-t)^{\gamma+1}}dt,$$

where $\Re\alpha+\gamma, \Re\beta+\gamma > -1$.

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   where $\Re\alpha+\gamma, \Re\beta+\gamma > -1$.

3. Decompose $\gamma = \delta + k$, $k \in \mathbb{N}_0$, then

$$Q_\gamma^{(\alpha,\beta)}(z) = \frac{(-1)^k k!}{2^{\gamma+1-k}(-\gamma)_k(z-1)^{\alpha}(z+1)^{\beta}} \int_{-1}^{1} \frac{(1-t)^{\alpha+\gamma-k}(1+t)^{\beta+\gamma-k}}{(z-t)^{\gamma-k+1}} P_k^{(\alpha+\gamma-k,\beta+\gamma-k)}(t)\, dt$$

Main theoretical result:

the multiple integral identity

Lemma. The multiple integral identity

Let $n, r \in \mathbb{N}_0$, $a, x \in \mathbb{C}$, $\vec{\mu} \in \mathbb{C}^r$, and let $f^{\vec{\mu}}$ be a function such that

$$\frac{d}{dz} f^{\vec{\mu}}(z) = \lambda_{\vec{\mu}} f^{\vec{\mu} \pm \vec{1}}(z),$$

where $\lambda_{\vec{\mu}} \in \mathbb{C}^*$. Then, the following identity holds:

$$\int_a^x \cdots \int_a^x f^{\mu}(w)(dw)^n = \frac{1}{\lambda_{\vec{\mu}\mp\vec{1}} \cdots \lambda_{\vec{\mu}\mp n\vec{1}}} \sum_{k=n}^{\infty} \lambda_{\vec{\mu}\mp n\vec{1}} \cdots \lambda_{\vec{\mu}\mp n\vec{1}\pm(k-1)\vec{1}} \, f^{\vec{\mu}\mp n\vec{1}\pm k\vec{1}}(a) \frac{(x-a)^k}{k!},$$

where $\vec{1} = (1, 1, ..., 1) \in \mathbb{C}^r$.

The Forward Shift Operator

1. $$\frac{d^n}{dz^n}(1-z)^{\alpha}(1+z)^{\beta}P_{\gamma}^{(\alpha,\beta)}(z) = (-2)^n(\gamma+1)_n(1-z)^{\alpha-n}(1+z)^{\beta-n}P_{\gamma+n}^{(\alpha-n,\beta-n)}(z)$$

$$\frac{d^n}{dz^n}(z-1)^{\alpha}(1+z)^{\beta}Q_{\gamma}^{(\alpha,\beta)}(z) = (-2)^n(\gamma+1)_n(z-1)^{\alpha-n}(1+z)^{\beta-n}Q_{\gamma+n}^{(\alpha-n,\beta-n)}(z).$$

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$$\frac{d^n}{dz^n}(z-1)^\alpha(1+z)^\beta Q_\gamma^{(\alpha,\beta)}(z) = (-2)^n(\gamma+1)_n(z-1)^{\alpha-n}(1+z)^{\beta-n}Q_{\gamma+n}^{(\alpha-n,\beta-n)}(z).$$

2. $$\frac{d^n}{dz^n}(1-z)^\alpha P_\gamma^{(\alpha,\beta)}(z) = (-\alpha-\gamma)_n(1-z)^{\alpha-n}P_\gamma^{(\alpha-n,\beta+n)}(z)$$
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3. $$\frac{d^n}{dz^n}(1+z)^{\beta}P_{\gamma}^{(\alpha,\beta)}(z) = (-1)^n(-\beta-\gamma)_n(1+z)^{\beta-n}P_{\gamma}^{(\alpha+n,\beta-n)}(z)$$

$$\frac{d^n}{dz^n}P_{\gamma}^{(\alpha,\beta)}(z) = 2^{-n}(\alpha+\beta+\gamma+1)_nP_{\gamma-n}^{(\alpha+n,\beta+n)}(z)$$

A Rodrigues formula for the Jacobi functions of the second kind

Starting with

$$\frac{d^n}{dz^n}(z-1)^{\alpha}(1+z)^{\beta}Q_{\gamma}^{(\alpha,\beta)}(z)=(-2)^n(\gamma+1)_n(z-1)^{\alpha-n}(1+z)^{\beta-n}Q_{\gamma+n}^{(\alpha-n,\beta-n)}(z),$$

mapping $(\alpha,\beta,\gamma)\to(\alpha-n,\beta-n,\gamma-n)$ one gets

$$Q_{\gamma}^{(\alpha,\beta)}(z)=\frac{2^{-n}}{(-\gamma)_n(z-1)^{\alpha}(z+1)^{\beta}}\frac{d^n}{dz^n}(z-1)^{\alpha+n}(z+1)^{\beta+n}Q_{\gamma-n}^{(\alpha+n,\beta+n)}(z).$$

Part II

The multi-integral representations

The $w(z) = (1-z)^\alpha(1+z)^\beta$ case

The Jacobi functions of the first kind

Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, such that $\alpha + \gamma \notin -\mathbb{N}$, $\Re\alpha$, $\Re\beta > -1$, $(-\gamma)_n \neq 0$, $z \in \mathbb{C} \setminus (-\infty, -1]$. Then

$$\int_z^1 \cdots \int_z^1 (1-w)^\alpha (1+w)^\beta P_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{(-1)^n}{2^n(-\gamma)_n}(1-z)^{\alpha+n}(1+z)^{\beta+n} P_{\gamma-n}^{(\alpha+n,\beta+n)}(z).$$

The Jacobi functions of the second kind

Let $n \in \mathbb{N}_0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $z \in \mathbb{C} \setminus [-1, 1]$, with $\Re\alpha$, $\Re\beta > -1$, $\Re\gamma > n$. Then

$$\int_z^\infty \cdots \int_z^\infty (w-1)^\alpha (1+w)^\beta Q_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{(z-1)^{\alpha+n}(1+z)^{\beta+n}}{2^n(\gamma-n+1)_n} Q_{\gamma-n}^{(\alpha+n,\beta+n)}(z).$$

The $w(z) = (1-z)^\alpha$ case

The Jacobi functions of the first kind

If $\Re\alpha > -1$, $z \in \mathbb{C} \setminus (-\infty, -1]$. Then

$$\int_z^1 \cdots \int_z^1 (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{(1-z)^{\alpha+n}}{(\alpha+\gamma+1)_n} P_\gamma^{(\alpha+n,\beta-n)}(z).$$

If $\Re(\alpha+\gamma) < -n$, $\Re(\beta+\gamma) > n-1$. Then

$$\int_z^\infty \cdots \int_z^\infty (1-w)^\alpha P_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{(1-z)^{\alpha+n}}{(\alpha+\gamma+1)_n} P_\gamma^{(\alpha+n,\beta-n)}(z).$$

The Jacobi functions of the second kind

If $\Re\alpha > -1$, $\Re\beta > n-1$, $\Re(\beta+\gamma+1) > n$, and $z \in \mathbb{C} \setminus [-1, 1]$. Then

$$\int_z^\infty \cdots \int_z^\infty (w-1)^\alpha Q_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{(z-1)^{\alpha+n}}{(\alpha+\gamma+1)_n} Q_\gamma^{(\alpha+n,\beta-n)}(z).$$

Lemma

If $\alpha+\gamma \notin -\mathbb{N}$, $z \in \mathbb{C} \setminus (-\infty,-1]$. *Then*

$$\int_z^1 \cdots \int_z^1 P_\gamma^{(\alpha,\beta)}(w)\,(dw)^n = \frac{2^n}{(-\alpha-\beta-\gamma)_n} P_{\gamma+n}^{(\alpha-n,\beta-n)}(z)$$

$$+ \frac{2\Gamma(\alpha+\gamma+1)(1-z)^{n-1}}{(n-1)!(\alpha+\beta+\gamma)\Gamma(\alpha)\Gamma(\gamma+2)}\, {}_3F_2\left({1-n, 1-\alpha, 1 \atop \gamma+2, 1-\alpha-\beta-\gamma} ; \frac{2}{1-z} \right)$$

$$= \frac{\Gamma(\alpha+\gamma+1)(1-z)^n}{\Gamma(\alpha+1)\Gamma(\gamma+1)n!}\, {}_3F_2\left({-\gamma, \alpha+\beta+\gamma+1, 1 \atop \alpha+1, n+1} ; \frac{1-z}{2} \right)$$

The Jacobi functions of the first kind

If $\alpha + \gamma \notin -\mathbb{N}$, $z \in \mathbb{C} \setminus (-\infty, -1]$. Then

$$\left[(z-1)^2 \frac{d}{dz}\right]^n (z-1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(z) = (\alpha+\beta+\gamma+1)_n (z-1)^{\alpha+\beta+\gamma+1+n} P_\gamma^{(\alpha,\beta+n)}(z),$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n \frac{1}{(z-1)^\gamma} P_\gamma^{(\alpha,\beta)}(z) = \frac{(-\alpha-\gamma)_n}{(z-1)^{\gamma-n}} P_{\gamma-n}^{(\alpha,\beta+n)}(z),$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n (z+1)^\beta (z-1)^{\alpha+\gamma+1} P_\gamma^{(\alpha,\beta)}(z)$$
$$= 2^n (\gamma+1)_n (z+1)^{\beta-n} (z-1)^{\alpha+\gamma+1+n} P_{\gamma+n}^{(\alpha,\beta-n)}(z),$$

$$\left[(z-1)^2 \frac{d}{dz}\right]^n \frac{(z+1)^\beta}{(z-1)^{\beta+\gamma}} P_\gamma^{(\alpha,\beta)}(z) = 2^n (-\beta-\gamma)_n \frac{(z+1)^{\beta-n}}{(z-1)^{\beta-n+\gamma}} P_\gamma^{(\alpha,\beta-n)}(z).$$

A Rodrigues formula for the Jacobi functions of the first kind

Starting with

$$\left[(z-1)^2\frac{d}{dz}\right]^n (z{+}1)^{\beta}(z{-}1)^{\alpha+\gamma+1}P_{\gamma}^{(\alpha,\beta)}(z) = 2^n(\gamma{+}1)_n(z{+}1)^{\beta-n}(z{-}1)^{\alpha+\gamma+1+n}P_{\gamma+n}^{(\alpha,\beta-n)}(z).$$

mapping $(\beta,\gamma)\to(\beta+n,0)$ one gets

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!}\frac{1}{(z-1)^{\alpha+n+1}(z+1)^{\beta}}\left[(z-1)^2\frac{d}{dz}\right]^n(z{-}1)^{\alpha+1}(z{+}1)^{\beta+n}.$$

More multi-integral representations

if $\alpha+\gamma\notin-\mathbb{N}$, $z\in\mathbb{C}\setminus(-\infty,-1]$. Then

$$\int_1^z\cdots\int_1^z (w-1)^{\alpha+\beta+\gamma+1}P_\gamma^{(\alpha,\beta)}(w)[(w-1)^{-2}\,\mathrm{d}w]^n=\frac{(z-1)^{\alpha+\beta+\gamma+1-n}}{(\alpha+\beta+\gamma-n+1)_n}P_\gamma^{(\alpha,\beta-n)}(z),$$

$$\int_1^z\cdots\int_1^z (w+1)^{\beta}(w-1)^{\alpha+\gamma+1}P_\gamma^{(\alpha,\beta)}(w)[(w-1)^{-2}\,\mathrm{d}w]^n=\frac{(z+1)^{\beta+n}(z-1)^{\alpha+\gamma-n+1}}{2^n(\gamma-n+1)_n}P_{\gamma-n}^{(\alpha,\beta+n)}(z),$$

$$\int_1^z\cdots\int_1^z (w-1)^{\alpha}(w+1)^{\beta+\gamma+1}P_\gamma^{(\alpha,\beta)}(w)[(w+1)^{-2}\,\mathrm{d}w]^n=\frac{(z-1)^{\alpha+n}(z+1)^{\beta+\gamma-n+1}}{2^n(\gamma-n+1)_n}P_{\gamma-n}^{(\alpha+n,\beta)}(z),$$

$$\int_1^z\cdots\int_1^z \frac{(w-1)^{\alpha}}{(w+1)^{\alpha+\gamma}}P_\gamma^{(\alpha,\beta)}(w)[(w+1)^{-2}\,\mathrm{d}w]^n=\frac{(z-1)^{\alpha+n}}{2^n(1+\alpha+\gamma)_n(z+1)^{\alpha+n+\gamma}}P_\gamma^{(\alpha+n,\beta)}(z),$$

where $\Re(\alpha+\beta+\gamma+1)>n$, $\Re(\alpha+\gamma+1)>n$, $\Re(\alpha+n)>0$, $\Re(\alpha+n)>0$, respectively.

If $z \in \mathbb{C} \setminus (-\infty, -1]$. Then

$$\int_1^z \cdots \int_1^z (w+1)^{\alpha+\beta+\gamma+1} P_\gamma^{(\alpha,\beta)}(w)[(w+1)^{-2}\,\mathrm{d}w]^n = \frac{(z+1)^{\alpha+\beta+\gamma+1-n} P_\gamma^{(\alpha-n,\beta)}(z)}{(\alpha+\beta+\gamma+1-n)_n}$$

$$-\frac{2^{\alpha+\beta+\gamma+1-n}\Gamma(\alpha+\gamma)}{(\alpha+\beta+\gamma)\Gamma(\alpha,\gamma+1,n)}\left(\frac{z-1}{z+1}\right)^{n-1} {}_3F_2\left(\begin{matrix}-n+1, 1-\alpha, 1\\ 1-\alpha-\gamma, 1-\alpha-\beta-\gamma\end{matrix}; \frac{z+1}{z-1}\right),$$

$$\int_1^z \cdots \int_1^z \frac{1}{(w+1)^\gamma} P_\gamma^{(\alpha,\beta)}(w)[(w+1)^{-2}\,\mathrm{d}w]^n = \frac{P_{\gamma+n}^{(\alpha-n,\beta)}(z)}{(\beta+\gamma+1)_n (z+1)^{\gamma+n}}$$

$$-\frac{\Gamma(\alpha+\gamma+1)}{2^{\gamma+n}(\beta+\gamma+1)\Gamma(\alpha,\gamma+2,n)}\left(\frac{z-1}{z+1}\right)^{n-1} {}_3F_2\left(\begin{matrix}-n+1, 1-\alpha, 1\\ 2+\gamma, 2+\beta+\gamma\end{matrix}; \frac{z+1}{z-1}\right).$$

A final remark

Note is of special interest the work by Loyal Durand [3] where the author derives many of the multi-integral representations appearing in [1] for associated Legendre and Ferrers functions from more general relations involving non-integer changes in the order obtained using fractional Lie group operator methods developed earlier for $\mathrm{SO}(2,1)$, $\mathrm{E}(2,1)$, and its conformal extension $\mathrm{SO}(3)$.

It is therefore probable that similar Lie group theoretic methods could be used to derive the multi-integral representations contained within the present paper (see [2]) for Jacobi functions of the first and second kind.

This could potentially shed some interesting light on the Lie groups which are associated with general Jacobi functions of the first and second kind, as well as these functions.

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THANK YOU for your attention!