Orthogonality of q-polynomials for non-standard parameters

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Abstract

q-polynomials can be defined for all the possible parameters but their orthogonality properties are unknown for several configurations of the parameters. Indeed, orthogonality for the Askey-Wilson polynomials, $p_n(x; a, b, c, d; q)$, is known only when none of the possible products of two of the four parameters a, b, c, d, is a negative integer power of q. Also, orthogonality for the big q-Jacobi polynomials, $p_n(x; a, b, c; q)$, is known when a, b, c, abc^{-1} are not a negative integer power of q. In this paper, we obtain orthogonality properties for the Askey-Wilson polynomials and the big q-Jacobi polynomials for the rest of the parameters and for all $n \in \mathbb{N}_0$. For a few values of such parameters, the three-term recurrence relation (TTRR)

$$xp_n = p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}\,, \qquad n \geq 0\,,$$

presents some index for which the coefficient $\gamma_n = 0$, and hence Favard's theorem can not be applied. For this purpose, we state a degenerate version of Favard's theorem, which is valid for all sequences of polynomials satisfying a TTRR even when some coefficient γ_n vanishes, i.e., $\{n : \gamma_n = 0\} \neq \emptyset$.

We also apply this result to the continuous dual q-Hahn, big q-Laguerre, q-Meixner, and little q-Jacobi polynomials, although it is also applicable to any family of orthogonal polynomials, in particular the classical orthogonal polynomials.

Keywords: q-Orthogonal polynomials, Favard's theorem, q-Hahn tableau, q-Askey tableau.
2000 MSC: Primary 42C05, 33C45, Secondary 33E30

1. Introduction

In the last two decades, some of the classical orthogonal polynomials with non-classical parameters have been provided with certain non-standard orthogonality properties (see, e.g., [1, 2, 3, 9, 13, 14, 16, 17, 18, 19, 20] and the references therein). The first case was given in 1995 by K. H. Kwon and L. L. Littlejohn [14], who established

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[1] RSCS acknowledges financial support from the Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01, and the program of postdoctoral grants (Programa de becas postdoctorales)
[2] JFSL acknowledges financial support from the Ministerio de Ciencia e Innovación of Spain, grant MTM2008-06689-C02, and Junta de Andalucía, grant FQM229.

the Sobolev orthogonality for the Laguerre polynomials $L_n^{(-N)}$ with $N \in \mathbb{N}$:

$$\langle f, g\rangle = \sum_{m=0}^{N-1}\sum_{j=0}^{m} B_{m,j}(N)\left(f^{(m)}(0)g^{(j)}(0) + f^{(j)}(0)g^{(m)}(0)\right) + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}dx\,, \tag{1}$$

with

$$B_{m,j}(N) = \begin{cases} \displaystyle\sum_{p=0}^{j}(-1)^{m+j}\binom{N-1-p}{m-p}\binom{N-1-p}{j-p} & \text{for} \quad 0 \le j < m \le N-1\,, \\ \displaystyle\frac{1}{2}\sum_{p=0}^{m}\binom{N-1-p}{m-p}^2 & \text{for} \quad 0 \le j = m \le N-1\,. \end{cases}$$

In 1998, M. Álvarez de Morales et al. [3] found, using a different technique, the orthogonality for the Gegenbauer polynomials $C_n^{(-N+1/2)}$. The case of Jacobi polynomials with negative integer parameters was studied by T. E. Pérez and M. A. Piñar et al. [1] and [2]. In these cases, the orthogonality can be written in the form

$$\langle f, g\rangle = F^t AG + \int f^{(N)}(x)g^{(N)}(x)d\mu(x)\,, \tag{2}$$

where F and G are vectors obtained by evaluating f and g and maybe their derivatives at some points, A is a symmetric real matrix, and μ is the orthogonality measure associated with the Nth derivative of the Laguerre, Gegenbauer, or Jacobi polynomials. Observe that the parameters considered in these cases are precisely those for which the three-term recurrence relation (TTRR)

$$xp_n = p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}\,, \qquad n \ge 0\,, \tag{3}$$

has one coefficient, $\gamma_N = 0$. The first term in the inner product (2) plays the role of the orthogonality for polynomials of degree lower than N. In this term, the points for the evaluation are the roots of p_N; this ensures that it vanishes also when one entry is p_n with $n \ge N$, since p_N is a factor of any p_n with $n \ge N$. The second term is relevant for polynomials with degree greater than N, and it is needed in order to have an orthogonality characterizing all the sequence $(p_n)_{n=0}^{\infty}$. The technique used in [3] is applicable for all classical orthogonal polynomials with one coefficient γ_n vanishing, and in fact it has been used recently in [16, 17, 18, 19].

In 2009, we considered the Racah, Hahn, dual Hahn, and Krawtchouk polynomials for all positive integer degrees. The corresponding TTRR for these polynomials also presents one vanishing coefficient for $n = N$, i.e., $\gamma_N = 0$, and in such a case the orthogonality found can be written as

$$\langle f, g\rangle = \int f(x)g(x)d\mu_d(x) + \int f^{(N)}(x)g^{(N)}(x)d\mu_c(x)\,, \tag{4}$$

where μ_d is a discrete measure with a finite number of masses and μ_c is an absolute continuous measure. For more details, see [9].

Although the two bilinear forms (2) and (4) are similar, they are not exactly of the same type. The part $F^t AG$ in (2) is expressed as a bilinear functional while the part with μ_d in (4) can be expressed as a linear functional. For any sequence (p_n) (finite or infinite) of normal polynomials (p_n) , i.e., $\deg p_n = n$ for all n, the existence of a linear functional, $\mathscr{L}$, such that the orthogonality conditions

$$\mathscr{L}(p_n p_m) = K_n \delta_{n,m}\,, \qquad K_n \neq 0\,,$$

characterize the sequence (p_n), is equivalent to the existence of the TTRR (3) with no vanishing coefficient γ_n. On the other hand, any family of normal polynomials (p_n) can be provided with an orthogonality given by a bilinear functional and characterizing the sequence (p_n). This bilinear functional is defined as

$$\langle p_n, p_m \rangle = K_n \delta_{n,m}\,, \tag{5}$$

with arbitrary K_n, with the exception $K_n \neq 0$ in general and $K_n > 0$ if we look for an inner product. Thus, if an orthogonality relation is given as a bilinear functional, it does not reveal the TTRR unless it is Hankel type, i.e.,

$$\langle xf, g \rangle = \langle f, xg \rangle\,,$$

and then it could be expressed as a linear functional. In fact, the part $F^t AG$ in (2) is obtained as the matrix form of (5) for $(p_n)_{n=0}^{N}$ with an adequate change of basis and as a consequence this part includes non-explicit terms. On the other hand, the measure μ_d in (4) is totally explicit for the families studied in [9].

The aim of this paper is to develop the ideas from [9] in order to obtain a general tool which provides a non-standard orthogonality property of the form (4) for any sequence of classical orthogonal polynomials with some vanishing coefficient γ_n. The tool is a degenerate version of the Favard theorem, developed in Section 2, which is also valid when there exist more than one index n so that $\gamma_n = 0$ and even when the family of orthogonal polynomials is not classical. Furthermore, with the help of this technique, we study in detail the orthogonality properties of the Askey-Wilson (Section 3) and big q-Jacobi (Section 4) polynomials for almost all values of the parameters. We also introduce in Section 5 the orthogonality properties for the continuous dual q-Hahn, big q-Laguerre, q-Meixner, and little q-Jacobi polynomials.

Notice that for the abbreviations in the tables we use a reduced notation taking into account [12], for example, AW means Askey-Wilson polynomials, cd$q$H means continuous dual q-Hahn polynomials, QqK means quantum q-Krawtchouk polynomials, qM means q-Meixner polynomials, and so forth.

2. Degenerate version of Favard's theorem

First, we recall some concepts and results concerning orthogonal polynomial sequences.

Definition 2.1. *A polynomial sequence $(p_n)_{n\in\mathbb{N}_0}$ with $\deg p_n = n$ is said to be an orthogonal polynomial system (OPS) with respect to a bilinear functional $\langle\cdot,\cdot\rangle$ if*

$$\langle p_n, p_m \rangle = K_n \delta_{n,m},$$

with $K_n \neq 0$ for all $n \in \mathbb{N}_0$.

The condition $K_n \neq 0$ guarantees that, given the bilinear functional, if an OPS exists then it is unique apart from the different normalizations, i.e., the sequence (p_n) is characterized by the bilinear functional through the conditions $\langle p_n, p_m \rangle \neq 0$ for all $n \neq m$.

When the bilinear functional can be represented as a linear functional, i.e., $\langle p, q \rangle = \mathscr{L}(pq)$, then it is known that the OPS satisfies a TTRR of the form (3) with $\gamma_n \neq 0$. Conversely, Favard's theorem asserts that if a family of polynomials satisfy the TTRR (3) with $\gamma_n \neq 0$ for all n, then the sequence of these polynomials is an OPS with respect to some linear functional $\mathscr{L}$. The aim of this section is to extend Favard's theorem for a TTRR with the possibility of some $\gamma_n = 0$ (for several considerations on Favard's theorem and its extensions, see [15]).

Consider the sequences (β_n) and (γ_n) of complex numbers and the family of monic polynomials generated by the following recurrence relation:

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x), \quad n = 1, 2, \ldots, \tag{6}$$

with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \beta_0$. By Favard's theorem, for $n \geq 0$, we define the moment functional as

$$\mathscr{L}_0(p_n) = \delta_{n,0}.$$

Notice that $\mathscr{L}_0(p_n p_m) = 0$ for all $n \neq m$ and in fact $\mathscr{L}_0$ is the unique moment functional satisfying these orthogonality conditions. Due to a result of R. P. Boas's [7], $\mathscr{L}_0$ can be represented as

$$\mathscr{L}_0(f) = \int_{-\infty}^{\infty} f(x) d\phi(x),$$

where ϕ is a complex function of bounded variation (see, for example, [8, §II, theorem 6.4]). Thus this orthogonality is in general non-hermitian, and its representation is not unique. For the cases we are going to study, a simpler representation will be obtained.

It is important to point out that if there exists N so that $\gamma_N = 0$, then, for $n \geq N$, $\mathscr{L}_0(x^j p_n) = 0$ for all $j \in \mathbb{N}_0$ (which also implies that $\mathscr{L}_0(p_N^2) = 0$) and thus the linear functional $\mathscr{L}_0$ does not determine the complete polynomial sequence (p_n) since, in particular,

$$\mathscr{L}_0\big((p_{N+1} + c p_N)x^m\big) = 0,$$

for all $m \leq N$ and for all $c \in \mathbb{C}$.

This means that if there exists $\gamma_N = 0$ then it is not possible to find an orthogonality of the form

$$\int p_n(x) p_m(x) d\mu(x) = K_n \delta_{n,m},$$

with μ a complex measure, characterizing the whole sequence (p_n), i.e., (p_n) is not an OPS with respect to $\mathscr{L}_0$. So the alternative is the use of a bilinear functional $\langle \cdot, \cdot \rangle$ such that

$$\langle p_n, p_m \rangle = K_n \delta_{n,m}\,,$$

with $K_n \neq 0$.

When the polynomials (p_n) are classical, the family formed with their derivatives, differences or q-differences (depending on the family) are also orthogonal, so they verify a TTRR of the form (6) but usually with different coefficients (the exception is the Hermite polynomials whose coefficients coincide). Another relevant fact is that, if the TTRR associated with the family (p_n) has some vanishing coefficient γ_n, then the vanishing coefficients γ_n associated with (p'_n), (Δp_n) or $(\mathscr{D}_q p_n)$ are shifted to a lower position or even disappear. This implies that, after taking derivatives, differences, or q-differences enough times, the resulting polynomials have no vanishing coefficient γ_n, and Favard's theorem is applicable. But these operators are not the unique ones that satisfy these properties of the TTRR and on the other hand the main families of orthogonal polynomials are not classical. Thus a general approach with operators not necessarily of differential type is interesting.

The previous ideas motivate the following construction. Let us consider the family of polynomials (p_n) defined by (6) with some $\gamma_n = 0$ and suppose the existence of a linear operator $\mathscr{T}_1 : \mathbb{P} \longrightarrow \mathbb{P}$, and a monic polynomial sequence $(p_{n,1})$ satisfying the following conditions.

1. $\deg(\mathscr{T}_1(p)) = \deg(p) - 1$ for any polynomial p (if p=const. it is considered that $\mathscr{T}_1(p) = 0$).
2. The polynomial sequence $(p_{n,1})$ is obtained by

$$p_{n-1,1} \stackrel{\text{def}}{=} \frac{\mathscr{T}_1(p_n)}{c_{n,1}}, \qquad n \geq 1,$$

where $c_{n,1}$ is the leading coefficient of $\mathscr{T}_1(x^n)$. The sequence $(p_{n,1})$ satisfies the TTRR

$$p_{n+1,1}(x) = (x - \beta_{n,1})p_{n,1}(x) - \gamma_{n,1}p_{n-1,1}(x), \qquad \text{for } n \in \mathbb{N}\,,$$

where the sequence $(\gamma_{n,1})$ is such that there exists a strictly increasing mapping

$$\lambda : \{n \in \mathbb{N} : \gamma_{n,1} = 0\} \longrightarrow \{n \in \mathbb{N} : \gamma_n = 0\}\,,$$

with $\lambda(n) > n$. In other words, in the step from the sequence (γ_n) to the sequence $(\gamma_{n,1})$, the vanishing positions jump down, or disappear, and the appearance of new vanishing positions is not allowed.

For many families of q-polynomials and their relative natural q-difference operator the condition for λ can be written as

$$\gamma_{n,1} = 0 \iff \gamma_{n+1} = 0, \tag{7}$$

for all $n \in \mathbb{N}$, but there are another polynomials with a different jump (for example, the Gegenbauer polynomials $C_n^{(-N+1/2)}$ for which $\gamma_{n,1} = 0 \iff \gamma_{n+2} = 0$). Under these hypotheses, $(p_{n,1})$ would be also a monic polynomials sequence and the existence of a moment functional, namely $\mathscr{L}_1$, such that $\mathscr{L}_1(p_{n,1}p_{m,1}) = 0$ is guaranteed. This procedure can be iterated k times, giving a sequence of operators $\mathscr{T}_k$, recurrence coefficients $(\beta_{n,k})$ and $(\gamma_{n,k})$, and moment functionals $\mathscr{L}_k$. Hence, if we denote by $\mathscr{T}^{(k)} \stackrel{\text{def}}{=} \mathscr{T}_k \circ \cdots \circ \mathscr{T}_1$, then

$$\begin{gathered} p_{n,k} = C_{n,k}\mathscr{T}^{(k)}(p_{n+k})\,, \qquad C_{n,k} = 1/(c_{n+k,1}\cdots c_{n+1,k}) \neq 0, \\ p_{n+1,k} = (x - \beta_{n,k})p_{n,k} - \gamma_{n,k}p_{n-1,k}, \qquad p_{0,k} = 1,\ p_{1,k} = x - \beta_{0,k}, \\ \mathscr{L}_k(p_{n,k}p_{m,k}) = 0, \quad n \neq m. \end{gathered} \tag{8}$$

The key fact of this procedure is that, if $\gamma_N = 0$, then, after taking $\mathscr{T}^{(N)}$, the sequence $(\gamma_{n,N})$ has lost at least one vanishing position.

Taking into account this construction, we can state the degenerate version of Favard's theorem for the case of only one vanishing coefficient γ_n.

Theorem 2.2. *Let (p_n) be a polynomial sequence satisfying the TTRR (6), and suppose that there exists a unique $N \in \mathbb{N}$ with $\gamma_N = 0$. Then (p_n) is a monic OPS with respect to the bilinear functional*

$$\langle p, r \rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{T}^{(N)}(p)\mathscr{T}^{(N)}(r)) \,. \tag{9}$$

Proof: The choice of $\mathscr{T}^{(N)}$ and its link with $\mathscr{L}_N$ guarantees that $\langle p_n, p_m \rangle = 0$ for all $n \neq m$. Hence, we only need to check the orthogonality conditions $\langle p_n, p_n \rangle \neq 0$ for all $n \geq 0$ in order to prove that (p_n) is a monic OPS.

If $n < N$ then, by hypothesis,

$$\langle p_n, p_n \rangle = \mathscr{L}_0(p_n^2) = \gamma_n \cdots \gamma_1 \neq 0 \,.$$

If $n \geq N$ then $\mathscr{L}_0(p_n^2) = 0$, so, taking into account (8),

$$\begin{aligned} \langle p_n, p_n \rangle = \; & \mathscr{L}_N(\mathscr{T}^{(N)}(p_n)\mathscr{T}^{(N)}(p_n)) = \frac{1}{C^2_{N,n-N}} \mathscr{L}_N(p_{n-N,N}p_{n-N,N}) \\ = \; & \frac{\gamma_{n-N,N}\gamma_{n-N-1,N} \cdots \gamma_{1,N}}{C^2_{n-N,N}} \neq 0 \,, \end{aligned}$$

since $\gamma_{n,N} \neq 0$ for all $n \in \mathbb{N}$, by construction. □

Remark: The orthogonality (9) is in general non-hermitian, but if the coefficients of the TTRR are real, $\gamma_1, \ldots, \gamma_{N-1} > 0$ and $\gamma_{n,N} > 0$ for all n, then it is an inner product.

Remark: Notice that, if there exists $N' < N$ such that $\gamma_{n,N'} > 0$, for all n, and $\gamma_1, \ldots, \gamma_{N-1} > 0$, then the value N in (9) can be replaced by N', and in such a case the proof of the statement is similar. Now, $\langle p_n, p_n \rangle$ depends on the linear functionals $\mathscr{L}_0$ and $\mathscr{L}_{N'}$, and, in this case, it is the sum of two positive terms which do not vanish simultaneously.

Corollary 2.3. *Let (p_n) be a polynomial sequence satisfying the TTRR (6), and let $\Lambda \stackrel{\text{def}}{=} \{n \in \mathbb{N} : \gamma_n = 0\}$. Then (p_n) is a monic OPS with respect to*

$$\langle p, r \rangle = \mathscr{L}_0(pr) + \sum_{k \in \mathscr{A}} \mathscr{L}_k(\mathscr{T}^{(k)}(p)\mathscr{T}^{(k)}(r)), \tag{10}$$

where $\mathscr{A} = \{N_0, N_1, \ldots\}$ with $N_{j+1} = N_j + \min\{n : \gamma_{n,N_j} = 0\}$.

The proof is straightforward taking into account the proof of theorem 2.2.

Remark: Observe that, if the set Λ is finite, then $\mathscr{A}$ is finite as well. Moreover, in (10), for any two polynomials there is always a finite number of non-vanishing terms, so the result of corollary 2.3 remains, even if the set $\mathscr{A}$ is infinite.

Among the operators $\mathscr{T}_k$ satisfying the imposed conditions one of the most natural, and non-trivial, is

$$\mathscr{T}_1(p)(x) \stackrel{\text{def}}{=} \mathscr{L}_0\left(\frac{p(t) - p(x)}{t - x}\right), \qquad p_{n-1,1} = \mathscr{T}_1(p_n) = p^{(1)}_{n-1},$$

where $\mathscr{L}_0$ acts on the variable t. This is due the fact that this operator commutes with the multiplication operator by x, and thus the coefficients of the TTRR are shifted by one, i.e., for $n \geq 1$,

$$p_{n+1}^{(1)}(x) = (x - \beta_{n+1})p_n^{(1)}(x) - \gamma_{n+1}p_{n-1}^{(1)}(x).$$

Notice that, although theorem 2.2 seems to be new, it has been used implicitly in [9] where the operator $\mathscr{T}_k$ is the forward difference operator Δ or another one related with Δ and it is applied to Racah, Hahn, dual Hahn, and Krawtchouk polynomials. In fact this result can be applied to the Laguerre polynomials $L_n^{(-N)}$, Gegenbauer polynomials $C_n^{(-N+1/2)}$ and Jacobi polynomials $P_n^{(\alpha,\beta)}$ with one parameter being a negative integer, by taking the operator $\mathscr{T}_k$ as the derivative operator. We are going to focus throughout this paper on the orthogonality properties of q-polynomials where the operator $\mathscr{T}_k$ is a q-difference type operator.

3. The Askey-Wilson polynomials

This family of q-polynomials, which were introduced by Askey and Wilson in [4], are located at the top of the q-Askey tableau. The monic Askey-Wilson polynomials can be written as a basic hypergeometric series [12]

$$p_n(x; a, b, c, d|q) = \frac{(ab; q)_n(ac; q)_n(ad; q)_n}{(2a)^n(abcdq^{n-1}; q)_n} {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right),$$

with $x = \cos\theta$. Moreover they fulfill, for $n \geq 0$, the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

where $\beta_n = (a + a^{-1} - A_n - C_n)/2$, and $\gamma_n = A_{n-1}C_n/4$, where

$$A_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$
$$C_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}.$$

Since $\Lambda = \{n \in \mathbb{N} : \gamma_n = 0\}$,

$$\Lambda = \emptyset \iff ab, ac, ad, bc, bd, cd \notin \Omega(q) \stackrel{\text{def}}{=} \{q^{-k} : k \in \mathbb{N}_0\}.$$

In what follows, we study the orthogonality for all possible parameters such that the polynomials are normal (i.e., $\deg p_n = n$); therefore $a, b, c, d \in \mathbb{C}$ but $abcd \notin \Omega(q)$.

3.1. The orthogonality conditions for $|q| < 1$

It is known that, if the parameters a, b, c, and d are real, or occur in complex conjugate pairs if complex, $\max\{|a|, |b|, |c|, d|\} < 1$, the family fulfills the orthogonality conditions [5]

$$\frac{1}{2\pi}\int_{-1}^{1} p_m(x)p_n(x)\frac{\omega(x)}{\sqrt{1-x^2}}dx = d_n^{2(AW)}\delta_{n,m}, \qquad n, m \geq 0, \tag{11}$$

where $d_n^{2(AW)}$ is the squared norm of the monic Askey-Wilson polynomial of degree n

$$d_n^{2(AW)} = \frac{(abcdq^{2n};q)_\infty}{4^n(abcdq^{n-1};q)_n(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}, \qquad (12)$$

and

$$\omega(x) = \left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2 = \frac{h(x,1)h(x,-1)h(x,q^{\frac{1}{2}})h(x,-q^{\frac{1}{2}})}{h(x,a)h(x,b)h(x,c)h(x,d)},$$

with

$$h(x,\alpha) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty}(1-2\alpha x q^k + \alpha^2 q^{2k}) = (\alpha e^{i\theta}, \alpha e^{-i\theta};q)_\infty, \quad x = \cos\theta.$$

The orthogonality conditions given in (11) are a particular case of the non-hermitian complex orthogonality conditions,

$$\int_\Gamma p_n\left(\frac{z+z^{-1}}{2}\right)p_m\left(\frac{z+z^{-1}}{2}\right)W(z)dz = d_n^{2(AW)}\delta_{n,m}\,, \qquad (13)$$

which were obtained by Askey and Wilson (see [4]), where

$$W(z) = \frac{1}{z}w\left(\frac{z+z^{-1}}{2}\right).$$

The poles of w are

$$\frac{\alpha q^k + (\alpha q^k)^{-1}}{2}, \quad \alpha = a,b,c,d, \qquad k \in \mathbb{N}_0\,,$$

therefore W has convergent poles, since $|q|<1$, at

$$aq^k, \quad bq^k, \quad cq^k, \quad dq^k, \qquad k \in \mathbb{N}_0,$$

and divergent poles at

$$a^{-1}q^{-k}, \quad b^{-1}q^{-k}, \quad c^{-1}q^{-k}, \quad d^{-1}q^{-k}, \qquad k \in \mathbb{N}_0.$$

The contour Γ is a curve separating the divergent poles from the convergent poles, encircling them only once. In fact, if the parameters satisfy $\max\{|a|,|b|,|c|,|d|\} < 1$, then Γ can be taken as the unit circle; otherwise, it is a deformation of the unit circle.

The poles can be separated only if

$$a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \Omega(q),$$

so in the following we focus our attention on when this does not occur. Notice that it may happen even when $\Lambda = \emptyset$ if $a^2,b^2,c^2,d^2 \in \Omega(q)$, so we first study this case.

Let us assume that $a^2 = q^{-M}$ with $M \in \mathbb{N}_0$ and $ab, ac, ad, bc, bd, cd \notin \Omega(q)$. Although the poles can not be separated, there is no γ_n vanishing in the TTRR, so we look for a simple reformulation of (13). The poles that can not be separated are

$$Z = \{q^{-M/2},\, q^{1-M/2}, \ldots,\, q^{M/2}\}, \quad \text{or} \quad Z = \{-q^{-M/2},\, -q^{1-M/2}, \ldots,\, -q^{M/2}\}.$$

Notice that, if some of these poles coincide with the generated by b, c, or d, then ab, ac, or $ad \in \Omega(q)$, which is not possible with the assumptions. Hence, Z has empty intersection with the rest of the poles of W.

We consider this case as the limit for $p_n(\cdot\,; \alpha, b, c, d; q)$ with $\alpha \to a$, so the poles of $W(\cdot\,; \alpha, b, c, d; q)$ can be separated adequately. Thus the orthogonality conditions (13) are valid, and can be expressed as

$$\int_{\Gamma_1' \cup \Gamma_2'} p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n^{2(AW)} \delta_{n,m}\,,$$

where the curves Γ_1' and Γ_2' separate the poles. Therefore, these curves can be deformed in order to obtain the integral through two curves, Γ_1 and Γ_2, such that they separate the convergent poles from the divergent ones, except for the poles in Z which stand between the two curves, with several residues added (see figure 1).

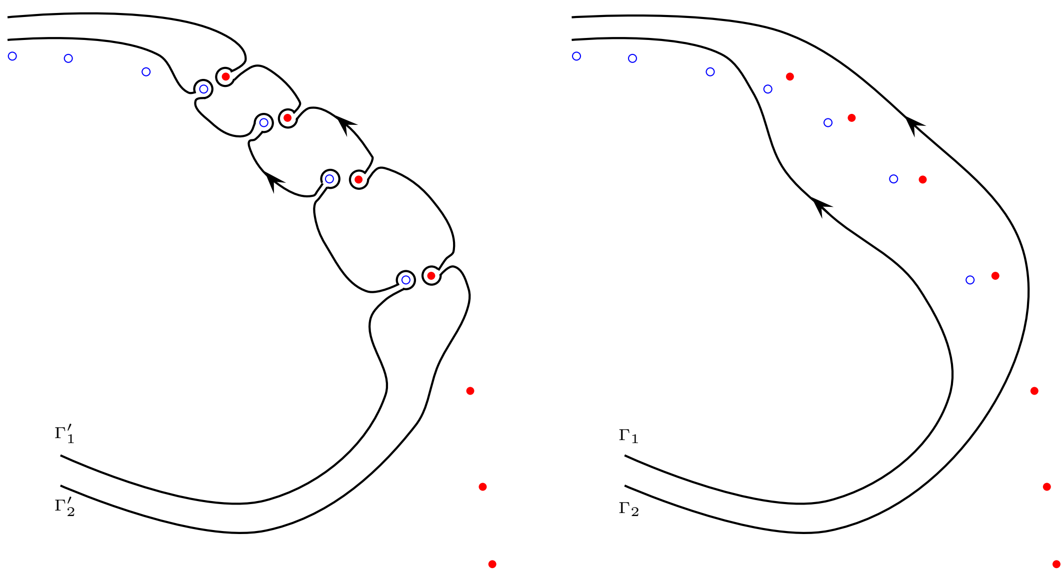

Figure 1: Transformation process to separate the poles.

When $\alpha \to a$, the poles αq^k with $k \le M$ and $\alpha^{-1} q^{-(M-k)}$ converge to aq^k, and it can be seen that the sum of the two residues at this points tends to zero. So, the limit $\alpha \to a$ yields

$$\int_{\Gamma_1 \cup \Gamma_2} p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n^{2(AW)} \delta_{n,m}, \tag{14}$$

with $d_n^{2(AW)}$ the normalizing factor given by (12).

We still have to study the orthogonality property of the Askey-Wilson polynomials in the case when $\Lambda \neq \emptyset$, i.e., when

$$ab, ac, ad, bc, bd, cd \in \Omega(q).$$

Furthermore, since any rearrangement of the parameters does not change the polynomial, it is enough to study the following two key cases.

- $ab = q^{-N+1}$, $ac, ad, bc, bd, cd \notin \Omega(q) \setminus \{q^{-N}\}$ and $a^2, b^2 \notin \{q^0, \ldots, q^{-N+2}\}$. This case corresponds to $\Lambda = \{N\}$, and p_N has only simple roots.

- $ab = q^{-N+1}$, $ac, ad, bc, bd, cd \notin \Omega(q) \setminus \{q^{-N}\}$ and $a^2 = q^{-M}$ with $M < N-1$, which corresponds to $\Lambda = \{N\}$ and p_N with some double roots.

All the other cases are combinations of these two cases, and the knowledge of (14) and corollary 2.3.

3.1.1.

$ab = q^{-N+1}$, $ac, ad, bc, bd, cd \notin \Omega(q) \setminus \{q^{-N}\}$ and $a^2, b^2 \notin \{q^0, \ldots, q^{-N+2}\}$; thus $\Lambda = \{N\}$.

Since monic q-Racah polynomials can be written in terms of the basic hypergeometric functions as [12]

$$r_n(\mu(x); \alpha, \beta, \gamma, \delta | q) = {}_4\varphi_3 \left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{array} \middle| q; q \right),$$

with $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$, the following identity linking Askey-Wilson and q-Racah polynomials holds:

$$p_n(x; a, b, c, d; q) = r_n(2ax; q^{-N}, cdq^{-1}, adq^{-1}, ad^{-1}; q),$$

and it yields the moment functional $\mathscr{L}_0$ in theorem 2.2 which is the one known for these q-Racah polynomials,

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2 q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1 - a^2 q^{2j})}{(cdq^{-N})^j (1 - a^2)} p\left(\frac{q^{-j} + a^2 q^j}{2a}\right).$$

Notice that the assumptions on a^2 and b^2 guarantee the definition of $\mathscr{L}_0$.

Remark: $\gamma_N = 0$ implies that p_N is a common factor of p_n for $n \geq N$; hence $\mathscr{L}_0(p_n) = 0$ also for $n \geq N$.

Furthermore,

$$\mathscr{D}_q p_n(x; a, b, c, d; q) = \frac{q^n - 1}{q - 1} p_{n-1}(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}; q),$$

where the q-difference operator, also called the Hahn operator, is

$$\mathscr{D}_q(f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \ \wedge \ q \neq 1, \\ f'(z), & z = 0 \ \vee \ q = 1. \end{cases}$$

This means that the family $(\mathscr{D}_q(p_n(\cdot; a, b, c, d)))$ also verifies a TTRR and it is easy to check that the position of the vanishing γ_n jumps down one position. Thus, the operator $\mathscr{T}$ can be chosen as $\mathscr{D}_q$, and conditions 1 and 2 for $\mathscr{T}$ hold. Hence, repeating the process N times, one obtains

$$\mathscr{D}_q^N p_n(x; a, b, c, d; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}(x; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q),$$

for $n \geq N$, whose TTRR has no vanishing γ_n. Furthermore, $\mathscr{L}_N$ is the moment functional associated with the Askey-Wilson polynomials with parameters $aq^{N/2}$, $bq^{N/2}$, $cq^{N/2}$, and $dq^{N/2}$, i.e.,

$$\mathscr{L}_N(p) = \int_\Gamma p\left(\frac{z + z^{-1}}{2}\right) \frac{1}{z} w\left(\frac{z + z^{-1}}{2}\right) dz,$$

where

$$w(z) = w(z; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q),$$

and Γ is a contour which separates the poles. Then, by theorem 2.2, the polynomial sequence $(p_n(x; a, b, c, d))$ is an OPS with respect to

$$\langle p, r\rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{D}_q^N(p)\mathscr{D}_q^N(r)).$$

3.1.2.

$ab = q^{-N+1}$, $ac, ad, bc, bd, cd \notin \Omega(q)\setminus\{q^{-N}\}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$.

The orthogonality is basically the same as that in 3.1.1, but now $\mathscr{L}_0$ is not valid since it has lost several orthogonality conditions. The adequate form of $\mathscr{L}_0$ is obtained as a limit case. Let us consider the linear functional

$$\mathscr{L}_0^{\alpha}(p) = \sum_{j=0}^{N-1} A_j(\alpha) p\left(\mu_j(\alpha)\right),$$

with $\mu_j(x; \alpha) = (\alpha q^j + \alpha^{-1} q^{-j})/2$, and

$$A_j(\alpha) = \frac{(q^{-N+1}, \alpha c, \alpha d, \alpha^2; q)_j}{(q, \alpha^2 q^N, \alpha c^{-1} q, \alpha d^{-1} q; q)_j} \frac{(1-\alpha^2 q^{2j})}{(cdq^{-N})^j(1-\alpha^2)}.$$

Straightforward computation yields

$$A_j(a) = 0,$$

for $j \in \{M+1, \ldots, N-1\}$ and $j = M/2$ if M is even, and

$$A_j(a) + A_{M-j}(a) = 0, \qquad \mu_j(a) = \mu_{M-j}(a),$$

for $j \in \{0, \ldots, M\}$, but $j = M/2$ if M is even. Thus $\mathscr{L}_0^{\alpha}$ tends to the null functional. But since it is possible to consider any normalization for $\mathscr{L}_0^{\alpha}$, we remove the common factor $(\alpha - a)$,

$$\lim_{\alpha\to a} \frac{A_j(\alpha)}{\alpha - a} p(\mu_j(\alpha)) = A_j'(a) p(\mu(a)),$$

for $j = M+1, \ldots, N$, and if M is even $j = M/2$, and also

$$\lim_{\alpha\to a} \frac{A_j(\alpha)p(\mu_j(\alpha)) + A_{M-j}(\alpha)p(\mu_{M-j}(\alpha))}{\alpha - a} = (A_j'(a) + A_{M-j}'(a))p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a)),$$

for $j = 0, \ldots, M$, except if M is even, $j \neq M/2$.

Hence we define $\mathscr{L}_0$ as

$$\mathscr{L}_0(p) = \sum_{j=0}^{(M-1)/2} \left(A_j'(a) + A_{M-j}'(a)\right)p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a) + \sum_{j=M+1}^{N-1} A_j'(a) p(\mu_j(a)),$$

if M is odd, and

$$\mathscr{L}_0(p) = \sum_{j=0}^{M/2-1} (A'_j(a) + A'_{M-j}(a))p(\mu_j(a)) + A_j(a)(q^j - q^{M-j})p'(\mu_j(a)) \\ + \sum_{j=M+1}^{N-1} A'_j(a)p(\mu_j(a)) + A'_{M/2}(a)p(\mu_{M/2}(a)),$$

if M is even. Notice that p_N has simple roots on $\mu_j(a)$, $j = M+1, \dots, N$ and on $\mu_{M/2}(a)$ if M is even; the rest, $\mu_j(a)$, $j = 0, \dots, [(M-1)/2]$ are double roots.

Finally, the family of Askey-Wilson polynomials with parameters $ab = q^{-N+1}$, $a^2 = q^{-M}$, where $M = 0, \dots N-2$ and $ac, ad, bc, bd, cd \notin \Omega(q)\backslash\{q^{-N}\}$, is an OPS with respect to

$$\langle p, r\rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{D}_q^N(p)\mathscr{D}_q^N(r)),$$

where $\mathscr{L}_N$ is the same as that in section 3.1.1.

3.2. The orthogonality conditions for $|q| \geq 1$

Taking into account the relation between basic hypergeometric series [12]

$$ {}_4\varphi_3\left(\begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \,\middle|\, q; q\right) = {}_4\varphi_3\left(\begin{array}{c} q^{n}, a^{-1}, b^{-1}, c^{-1} \\ d^{-1}, e^{-1}, f^{-1} \end{array} \,\middle|\, q^{-1}; \frac{abcq^{-n}}{def}\right), \tag{15}$$

we can relate each family of q-polynomials on the parameter q into another family of q-polynomials on the parameter q^{-1}. In fact, in this case,

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q).$$

Therefore, if $|q| > 1$, we can get analogous orthogonality conditions just using this relation and the orthogonality conditions given in Section 3.1 for $|q| < 1$.

If q is a primitive root of unity, i.e., $q = e^{2\pi i M/N}$ with $\gcd(N, M) = 1$, then $\{kN : k \in \mathbb{N}\} \subseteq \Lambda$, so, by corollary 2.3, we need to construct the following orthogonality property for the Askey-Wilson polynomials up to degree N [6], i.e.,

$$\sum_{s=0}^{N-1} p_n(x_s)p_m(x_s)\omega_s = \gamma_1 \cdots \gamma_n \delta_{n,m},$$

where $n, m = 0, 1, \dots, N-1$, $\{x_s\}_{s=0}^{N-1}$ are the zeros of p_N, and the weight function is

$$\omega_s = \frac{\gamma_1 \cdots \gamma_{N-1}}{p_{N-1}(x_s)p'_N(x_s)}.$$

So the only requirement to be added is that all zeros x_s must be simple.

Since the method considered in [21] to obtain ω_s can be applied to obtain such weights functions for other families of q-polynomials, next we give a brief outline of it.

It is known that Askey-Wilson polynomials are polynomial eigenfunctions of the second-order homogeneous linear difference operator:

$$\sigma(-s)\frac{\Delta p_n(x(s))}{\Delta x(s)} + \sigma(s)\frac{\nabla p_n(x(s))}{\nabla x(s)} - \lambda_n \Delta x(s - \tfrac{1}{2})p_n(x(s)) = 0,$$

where $\sigma(s) = -(q^{1/2} - q^{-1/2})^2 q^{-2s+1/2}(q^s - a)(q^s - b)(q^s - c)(q^s - d)$, and their corresponding eigenvalues are

$$\lambda_n = -4q^{-n+1}(1 - q^n)(1 - abcdq^{n-1}).$$

Notice that such a difference operator can be rewritten, by using the definition of the difference operators Δ and ∇, (see [12], [21, Eq. (3.6)]) as

$$A(z^{-1})p_n(q^{-1}z) - (A(z) + A(z^{-1}))p_n(z) + A(z)p_n(qz) = \lambda_n p_n(z),$$

where $A(z) = (1-az)(1-bz)(1-cz)(1-dz)/((1-z^2)(1-qz^2))$. Therefore, multiplying the previous equation by a function $\rho(s)$ satisfying the only requirement of periodicity $\rho(s+N) = \rho(s)$, and combining it with a similar equation for the polynomials $p_m(x_s)$, one can get a bilinear relation:

$$\begin{gathered} A_s\sigma(s)\left(p_n(x_{s-1})p_m(x_s) - p_n(x_s)p_m(x_{s-1})\right) \\ +C_s\sigma(s)\left(p_n(x_{s+1})p_m(x_s) - p_n(x_s)p_m(x_{s+1})\right) \\ = (\lambda_n - \lambda_m)\sigma(s)p_n(x_s)p_m(x_s). \end{gathered}$$

Choosing $\rho(s)$ in such a way that

$$A_{s+1}\rho(s+1) = C_s\rho(s), \tag{16}$$

summing from $s = 0$ to $s = N-1$, and using the obvious periodicity property of $\rho(s)$, we get the orthogonality property:

$$(\lambda_n - \lambda_m)\sum_{s=0}^{N-1} p_n(x_s)p_m(x_s)\rho(s) = 0, \qquad n \neq m.$$

Hence, $\omega_s = \omega_0\rho(s)$, with ω_0 the normalization constant, is determined from relation (16).

With this approach, Spiridonov and Zhedanov found that the family of polynomials $(p_n(\cdot\,; a, b, c, d; e^{2\pi i M/N}))$ with $0 \le n \le N$ and the assumptions

$$abcd, ab, ac, ad, bc, bd, cd \neq q^k, \qquad k = 0, \dots, N-1\,,$$

is uniquely determined by the orthogonality conditions

$$\mathscr{L}_0(p_n p_m) = d_n^2\delta_{n,m}, \quad d_n^2 \neq 0,$$

where

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1}\left(\frac{q}{abcd}\right)^j \frac{(1-\tau q^{2j})(a\tau, b\tau, c\tau, d\tau; q)_j}{(1-\tau^2)(q\tau/a, q\tau/b, q\tau/c, q\tau/d; q)_j} p\left(\tau q^j + \tau^{-1}q^{-j}\right),$$

and τ the root with minimal argument of the equation

$$\tau^N = E_N/2 + \sqrt{E_N^2/4 - 1},$$

where

$$E_N = \frac{a^N + b^N + c^N + d^N - (abc)^N - (abd)^N - (acd)^N - (bcd)^N}{1 - (abcd)^N}.$$

Remark: A straightforward computation shows that

$$\rho(s) \stackrel{\text{def}}{=} \left(\frac{q}{abcd}\right)^s \frac{(1-\tau q^{2s})(a\tau, b\tau, c\tau, d\tau; q)_s}{(1-\tau^2)(q\tau/a, q\tau/b, q\tau/c, q\tau/d; q)_s},$$

satisfies condition (16). A hint for such calculation can be found in [10, Lemma 5.1].

Due to the cyclic behavior of the TTRR coefficients, and since $\gamma_N = 0$, these polynomials satisfy the identity

$$p_n = p_N^\ell p_m, \qquad n = \ell N + m, \qquad 0 \le m < N,$$

which explains the behavior of the polynomial for greater degrees. However corollary 2.3 is applicable. For $n \ge N$,

$$\mathscr{D}_q^N p_n(x; a, b, c, d; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}((-1)^M x; a, b, c, d; q),$$

so the orthogonality conditions that characterizes all polynomials are the following.

- If M is even:
$$\langle p, r\rangle = \sum_{j=0}^{\infty} \mathscr{L}_0(\mathscr{D}_q^{Nj}(p)\mathscr{D}_q^{Nj}(r)).$$

- If M is odd:
$$\langle p, r\rangle = \sum_{j=0}^{\infty} \left(\mathscr{L}_0(\mathscr{D}_q^{2jN}(p)\mathscr{D}_q^{2jN}(r)) + \mathscr{L}_N(\mathscr{D}_q^{(2j+1)N}(p)\mathscr{D}_q^{(2j+1)N}(r))\right),$$

  with

$$\mathscr{L}_N(p) = \sum_{j=0}^{N-1} \left(\frac{q}{abcd}\right)^j \frac{(1-\tau q^{2j})(a\tau, b\tau, c\tau, d\tau; q)_j}{(1-\tau^2)(q\tau/a, q\tau/b, q\tau/c, q\tau/d; q)_j} p\left(-\tau q^j - \tau^{-1} q^{-j}\right).$$

4. The big q-Jacobi polynomials

The big q-Jacobi polynomials, which were introduced by Hahn in 1949, are located at the top of the q-Hahn tableau. The monic big q-Jacobi polynomials can be written in terms of basic hypergeometric series as [12]

$$p_n(x; a, b, c; q) = \frac{(aq, cq; q)_n}{(abq^{n+1}; q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \,\middle|\, q; q\right). \tag{17}$$

In fact, they are the most general family of q-polynomials on the q-exponential lattice, also called the q-linear lattice; and they appear, among other branches of physics, in the representation theory of quantum algebras [22]. The monic big q-Jacobi polynomials fulfill, for $n \ge 1$, the following TTRR:

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{18}$$

with $\beta_n = 1 - \hat{A}_n - \hat{C}_n$, and $\gamma_n = \hat{A}_{n-1}\hat{C}_n$, where

$$\begin{aligned} \hat{A}_n &= \frac{(1-aq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \\ \hat{C}_n &= -acq^{n+1}\frac{(1-q^n)(1-abc^{-1}q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \end{aligned} \tag{19}$$

Remark: If $a = 0$ then the coefficients $\gamma_n = 0$ for all $n \in \mathbb{N}_0$, and, if $b = 0$, or $c = 0$, then the big q-Jacobi polynomials become the big q-Laguerre or the little q-Jacobi polynomials, respectively, which are located below in the q-Askey tableau, and thus we omit these cases.

A slightly less detailed study of orthogonality conditions for the big q-Jacobi polynomials can be found in [17], using a bilinear form instead of the linear form $\mathscr{L}_0$.

4.1. The orthogonality conditions for $|q| < 1$

It is known that, if $0 < q < 1$, $a, b \in (0, q^{-1})$, and $c < 0$ the family of big q-Jacobi polynomials fulfills the orthogonality conditions [12]

$$\int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} p_n(x; a, b, c; q) p_m(x; a, b, c; q) d_q x = d_n^{2(BqJ)} \delta_{n,m}, \tag{20}$$

where the Jackson q-integral (see [11, 12]) is defined as follows:

$$\int_a^b f(t) d_q t = a(q-1) \sum_{s=0}^{\infty} f(aq^s) q^s - b(q-1) \sum_{s=0}^{\infty} f(bq^s) q^s.$$

The aim of this Section is to give orthogonality conditions for the big q-Jacobi polynomials for general complex parameters, including complex $|q| < 1$, except for those for which the family is not normal, i.e., $ab \in \Omega(q)$.

In fact, notice that, if the parameters belong to compact sets where the integrand in (20) is bounded, such series converges uniformly. Thus we can apply the Weierstrass theorem and analytic prolongation in order to assert that (20) is valid for

$$a, b, c, abc^{-1} \notin \Omega(q),$$

which it is equivalent to $\Lambda = \emptyset$. Therefore in the following, we focus our attention on the case $\Lambda \neq \emptyset$. More precisely, we study the cases for which this set has exactly one element, namely N. If this set is greater, we refer the reader to corollary 2.3.

4.1.1.

$c = q^{-N}$ and $a, b, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$. Taking into account that the big q-Jacobi and q-Hahn polynomials are linked through the relation

$$p_n(x; a, b, q^{-N}; q) = h_n^{(a,b)}(x; N-1; q),$$

the moment functional $\mathscr{L}_0$ in theorem 2.2 is the one known for the q-Hahn polynomials [12] with parameters a, b and $N-1$:

$$\mathscr{L}_0(p) = \sum_{x=0}^{N-1} \frac{(aq, q^{-N+1}; q)_x}{(q, b^{-1}q^{-N+1}; q)_x} (abq)^{-x} \; p(q^{-x}). \tag{21}$$

Moreover, since

$$\mathscr{D}_{q^{-1}} h_n^{(a,b)}(x; M; q) = \frac{q^{-n}-1}{q^{-1}-1} h_{n-1}^{(qa,qb)}(x; M-1; q),$$

the operator $\mathscr{T}$ in theorem 2.2 can be chosen as $\mathscr{D}_{q^{-1}}$, and the condition (7) holds (see the relation between q-Hahn and big q-Jacobi polynomials and the expression (19) for the coefficients γ_n). Also, for $n \geq N$,

$$\mathscr{D}^N_{q^{-1}} p_n(x; a, b, q^{-N}; q) = \frac{(q^{-n}; q)_N}{(1-q^{-1})^N} p_{n-N}(x; aq^N, bq^N, 1; q).$$

Accordingly with these expressions and the weight function for the big q-Jacobi polynomials with parameters aq^N, bq^N and 1, if we take

$$\mathscr{L}_N(p) = \int_q^{aq^{N+1}} \frac{(a^{-1}q^{-N}x; q)_\infty}{(bq^N x; q)_\infty} p(x)\, d_q x,$$

then, by theorem 2.2, the mentioned big q-Jacobi polynomials are an OPS with respect to

$$\langle p, r\rangle = \mathscr{L}_0(pr) + \mathscr{L}_N(\mathscr{D}^N_{q^{-1}}(p)\mathscr{D}^N_{q^{-1}}(r)) \,.$$

4.1.2.

$a = q^{-N}$, $b, c, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$. Using the identity

$$p_n(x; a, b, c; q) = p_n(x; c, abc^{-1}, a; q), \tag{22}$$

which can be obtained easily from the hypergeometric representation (17) or from the TTRR (18), this case is reduced to that in section 4.1.1.

4.1.3.

$b = q^{-N}$ and $a, c, abc^{-1} \notin \Omega(q) \setminus \{q^{-N}\}$. The orthogonality in this case can be obtained taking the limit $b \to q^{-N}$.

Multiplying relation (20) by the factor $(b - q^{-N})$, taking the limit $b \to q^{-N}$, and removing some non-vanishing constants, one gets, for $n \neq m$,

$$\sum_{s'=0}^{N-1} \frac{(a^{-1}cq^{s'+1}, q^{s'+1}; q)_\infty\; q^{-N} q^{s'}}{(cq^{s'+1}; q)_\infty\; (q^{-N+s'+1}; q)_{N-s'-1}} p_n(cq^{s'+1}; a, q^{-N}, c; q) p_m(cq^{s'+1}; a, q^{-N}, c; q) = 0. \tag{23}$$

The others terms of the two series in the series representation for the Jackson q-integral vanish after taking the limit, since these series converge uniformly for b in a compact neighborhood of q^{-N}.

Reversing the summation and using the identity

$$(\alpha; q)_s = (\alpha^{-1}q^{1-s}; q)_s (-\alpha)^s q^{\binom{s}{2}},$$

orthogonality property (23) can be rewritten as

$$\sum_{s=0}^{N-1} \frac{(ac^{-1}q^{-N+1}, q^{-N+1}; q)_s}{(c^{-1}q^{-N+1}, q; q)_s} \frac{q^{(N-1)s}}{a^s} p_n(cq^N q^{-s}; a, q^{-N}, c; q) p_m(cq^N q^{-s}; a, q^{-N}, c; q) = 0. \tag{24}$$

Comparing (21) and (24), we get

$$\begin{aligned} p_n(x; a, q^{-N}, c; q) &= c^n q^{nN} h_n^{(ac^{-1}q^{-N}, c)}(c^{-1}q^{-N}x; N-1; q) \\ &= c^n q^{nN} p_n(c^{-1}q^{-N}x; ac^{-1}q^{-N}, c, q^{-N}; q). \end{aligned} \tag{25}$$

The identities used are not valid for several configurations of the parameters; however, (25) is also valid for these configurations by using analytic continuation. Thus the case treated in this subsection can be reduced to the case considered in subsection §4.1.1 by setting $x \mapsto c^{-1}q^{-N}x$.

It is curious that identity (25) has the hypergeometric form

$$
\begin{aligned}
&{}_3\varphi_2\left(\begin{matrix} q^{-n}, aq^{n-N+1}, x \\ aq, cq \end{matrix}\,\middle|\, q;q\right) \\
&= \frac{c^n q^{nN}(ac^{-1}q^{-N+1}, q^{-N+1};q)_n}{(aq,cq;q)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, aq^{n-N+1}, c^{-1}q^{-N}x \\ ac^{-1}q^{-N+1}, q^{-N+1} \end{matrix}\,\middle|\, q;q\right),
\end{aligned}
$$

which coincides with [11, (3.2.6)]

$$
{}_3\varphi_2\left(\begin{matrix} q^{-n}, \widehat{a}q^n, \widehat{b} \\ \widehat{d}, \widehat{e} \end{matrix}\,\middle|\, q; \frac{\widehat{d}\widehat{e}}{\widehat{a}\widehat{b}}\right) = \frac{(\widehat{a}q/\widehat{d}, \widehat{a}q/\widehat{e};q)_n}{(\widehat{d},\widehat{e};q)_n}\left(\frac{\widehat{d}\widehat{e}}{\widehat{a}q}\right)^n {}_3\varphi_2\left(\begin{matrix} q^{-n}, \widehat{a}q^n, \widehat{a}\widehat{b}q/\widehat{d}\widehat{e} \\ \widehat{a}q/\widehat{d}, \widehat{a}q/\widehat{e} \end{matrix}\,\middle|\, q; \frac{q}{\widehat{b}}\right),
$$

in the parameters but it does not in the arguments if one sets $\widehat{a} = aq^{-N+1}$, $\widehat{b} = x$, $\widehat{d} = aq$, and $\widehat{e} = cq$.

4.1.4.

$abc^{-1} = q^{-N}$ and $a, b, c \notin \Omega(q) \setminus \{q^{-N}\}$. Once again, by (22), this case can be reduced to the case in subsection 4.1.3.

4.2. The orthogonality conditions for $|q| \geq 1$.

Identities (15) and

$$
{}_3\varphi_2\left(\begin{matrix} q^{-n}, a, b \\ d, e \end{matrix}\,\middle|\, q;q\right) = \frac{(e/a;q)_n}{(e;q)_n}a^n\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, a, d/b \\ d, aq^{1-n}/e \end{matrix}\,\middle|\, q;\frac{bq}{e}\right),
$$

(see [11, (3.2.2)]), yield

$$
\begin{aligned}
&{}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q;q\right) \\
&= \frac{(c(abq)^{-1};q^{-1})_n}{(cq^n;q^{-1})_n}(ab)^n q^{n^2+n}\,{}_3\varphi_2\left(\begin{matrix} q^n, (ab)^{-1}q^{-n-1}, (aq)^{-1}x \\ (aq)^{-1}, c(abq)^{-1} \end{matrix}\,\middle|\, q^{-1};q^{-1}\right).
\end{aligned}
$$

which in terms of big q-Jacobi polynomials is written as

$$
p_n(x;a,b,c;q) = \frac{1}{(a^{-1}q^{-1})^n}\,p_n(a^{-1}q^{-1}x; a^{-1}, b^{-1}, ca^{-1}b^{-1}; q^{-1}).
$$

Hence, the orthogonality conditions for big q-Jacobi polynomials with $|q| > 1$ follow from Section 4.1.

If q is a primitive root of unity, i.e., $q = e^{2\pi i M/N}$ with $\gcd(N, M) = 1$, then $\{kN : k \in \mathbb{N}\} \subseteq \Lambda$, and as we did for the Askey-Wilson polynomials, the set of big q-Jacobi polynomials $(p_n(x;a,b,c;q))_{n=0}^N$ under the assumptions

$$
a, b, c, ab, abc^{-1} \neq q^k, \qquad k = 0, \ldots, N-1,
$$

is uniquely determined by the orthogonality conditions

$$
\mathscr{L}_0(p_n p_m) = d_n^2 \delta_{n,m}, \quad d_n^2 \neq 0,
$$

where

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \omega_0 \frac{(\tau a^{-1}q^j, \tau c^{-1}q^j; q)_\infty}{(\tau q^j, \tau bc^{-1}q^j; q)_\infty} q^s p(\tau q^j),$$

with ω_0 such that $\mathscr{L}_0(1) = 1$, and τ the root with minimal argument of the equation

$$\tau^N = \frac{a^N + c^N - (ab)^N - (ac)^N}{1 - (ab)^N}.$$

Moreover, since, for $n \geq N$,

$$\mathscr{D}_q^N p_n(x; a, b, c; q) = \frac{(q^{n-N+1}; q)_N}{(1-q)^N} p_{n-N}(x; a, b, c; q),$$

these big q-Jacobi polynomials are an OPS with respect to

$$\langle p, r \rangle = \sum_{j=0}^{\infty} \mathscr{L}_0(\mathscr{D}_q^{Nj}(p)\mathscr{D}_q^{Nj}(r)).$$

Remark: In [21], the particular case $c = 1$ is considered and, in such a case,

$$\omega_s = \frac{(1-a^N)(1-abq)(b;q)_s}{aq(b-1)(1-a^N b^N)(a^{-1};q)_s} q^s.$$

5. Nonstandard orthogonality properties for continuous dual q-Hahn, big q-Laguerre, q-Meixner, and little q-Jacobi polynomials for non classical parameters

One of the key points in obtaining the orthogonality properties is that, since $\gamma_N = 0$, the polynomials for degree n greater than N factorize as $p_n = p_N p_{n-N}^{(N)}$, where $(p_n^{(N)})$ is the family of Nth associated polynomials which fulfills, for $n \geq 0$, the recurrence relation

$$p_{n+1}^{(N)}(x) = (x - \beta_{n+N})p_n^{(N)}(x) - \gamma_{n+N} p_{n-1}^{(N)}(x),$$

with initial conditions $p_{-1}^{(N)}(x) \equiv 0$, $p_0^{(N)}(x) = 1$. This factorization is evident by using the TTRR, but what is not so evident is that the family of Nth associated polynomials is also orthogonal; in fact they are q-polynomials at the same level in the Askey tableau. Notice that in the case of q-polynomials the existence of an integer N so that $\gamma_N = 0$ is directly related with the existence of a term of the form q^{-N+1} in the denominator parameters of one of the hypergeometric representations. In such a case, the basic hypergeometric series ${}_p\varphi_{p-1}$ with a suitable normalization factorizes as follows.

Let $a = \{a_1, \ldots, a_{p-1}\}$ and $b = \{b_1, \ldots, b_{p-2}\}$; then

$$\begin{aligned}
&(q^{-N+1}; q)_{n+N}\ {}_p\varphi_{p-1}\left(\begin{matrix} q^{-n-N}, a \\ q^{-N+1}, b \end{matrix} \,\middle|\, q; z \right) \\
=&\frac{(q^{-n-N}; q)_N\,(a;q)_N z^N}{(b;q)_N (q;q)_N} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (aq^N;q)_k z^k}{(bq^N;q)_k (q^{N+1};q)_k}\ \frac{(q;q)_n}{(q;q)_k} \\
=&\frac{(q^{N+1};q)_n (a;q)_N z^N}{(b;q)_N} (-1)^N q^{N(-n-N)+N(N-1)/2}\ {}_p\varphi_{p-1}\left(\begin{matrix} q^{-n}, aq^N \\ q^{N+1}, bq^N \end{matrix} \,\middle|\, q; z \right).
\end{aligned}$$

Hence it is straightforward to obtain the following factorization:

$$\begin{aligned}(q^{-N+1};q)_{n+N}\ {}_p\varphi_{p-1}\left(\begin{matrix}q^{-n-N},a\\ q^{-N+1},b\end{matrix}\ \middle|\ q;z\right)&=q^{-nN}(q^{N+1};q)_n(q^{-N+1};q)_N\\ &\times{}_p\varphi_{p-1}\left(\begin{matrix}q^{-N},a\\ q^{-N+1},b\end{matrix}\ \middle|\ q;z\right){}_p\varphi_{p-1}\left(\begin{matrix}q^{-n},aq^N\\ q^{N+1},bq^N\end{matrix}\ \middle|\ q;z\right).\end{aligned}\tag{26}$$

Notice that the first hypergeometric function of the right-hand side of (26), with its corresponding normalization coefficient, is a polynomial of degree N, and the second one is the Nth associated polynomial in the factorization, so it also has a hypergeometric form. Table 1 shows the Nth associated polynomial as the corresponding q-polynomials where we follow the notation in [12].

Another key fact that lets us obtain explicit expressions for the orthogonality is that on the one hand Askey-Wilson and q-Racah polynomials and on the other hand big q-Jacobi and q-Hahn polynomials are essentially the same polynomials, the only difference being the form of the orthogonality. Identities of the same type also appears between continuous dual q-Hahn and dual q-Hahn, affine q-Krawtchouk and big q-Laguerre, quantum q-Krawtchouk and q-Meixner, and q-Krawtchouk and little q-Jacobi polynomials. Also, affine q-Krawtchouk and quantum q-Krawtchouk polynomials are the same polynomials after rescaling the variable. These identities can be seen in table 2. So we focus our attention on continuous dual q-Hahn, big q-Laguerre, q-Meixner, and little q-Jacobi polynomials. For all the families we consider a simple case, i.e., $\Lambda=\{N\}$, thus theorem 2.2 is applicable, and the orthogonality is

$$\langle p,r\rangle=\mathscr{L}_0(pr)+\mathscr{L}_N(\mathscr{T}^{(N)}(p)\mathscr{T}^{(N)}(r)).$$

5.1. Continuous dual q-Hahn

The monic continuous dual q-Hahn polynomials, $p_n(x;a,b,c|q)$, satisfy the TTRR

$$\begin{aligned}xp_n(x)&=p_{n+1}(x)+\frac{1}{2}(a+a^{-1}-(A_n+C_n))p_n(x)\\ &\quad+\frac{1}{4}(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-bcq^{n-1})p_{n-1}(x)\end{aligned}$$

with

$$A_n=a^{-1}(1-abq^n)(1-acq^n)\,,\qquad C_n=a(1-q^n)(1-bcq^{n-1})\,.$$

If we take $ac=q^{-N+1}$, with the help of table 2, we see that $\mathscr{L}_0$ is known for the dual q-Hahn $r_n(2ax;ab/q,a/b,N-1|q)$:

$$\mathscr{L}_0(p)=\sum_{x=0}^{N-1}\frac{(ab,a^2,q^{-N+1};q)_x}{(q,a^2q^N,aq/b;q)_x}\,\frac{(1-a^2q^{2x})}{(1-a^2)(-ab)^x}\,q^{(N-1)x-\binom{x}{2}}p\left(\frac{a^{-1}q^{-x}+aq^x}{2}\right).$$

The operator $\mathscr{T}$ is $\mathscr{D}_q$, since

$$\mathscr{D}_qp_n(x;a,b,q^{-N+1}/a|q)=const.p_{n-1}(x;aq^{1/2},bq^{1/2},q^{-N+3/2}|q),$$

which shifts down one position the vanishing coefficient γ_n of the new TTRR. Thus, after applying the linear operator $\mathscr{D}_q$ N times, one obtains the continuous dual q-Hahn polynomials $p_{n-N}(x;aq^{N/2},bq^{N/2},q^{-N+1+N/2}/a|q)$ for which $\Lambda=\emptyset$ and

$$\mathscr{L}_N(p)=\int_\Gamma p\left(\frac{z+z^{-1}}{2}\right)\frac{1}{z}w\left(\frac{z+z^{-1}}{2}\right)dz,$$

with

$$w(z) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,aq^{N/2})h(x,bq^{N/2})h(x,q^{1-N/2}/a)},$$
$$h(x,\alpha) = (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty, \qquad x = \cos\theta\,,$$

and Γ a contour separating the poles.

5.2. Big q-Laguerre

The monic big q-Laguerre polynomials, $p_n(x;a,b;q)$, satisfy the TTRR

$$\begin{aligned} xp_n(x) = p_{n+1}(x) + (1 - A_n - C_n)p_n(x) \\ - abq^{n+1}(1-q^n)(1-aq^n)(1-bq^n)p_{n-1}(x) \end{aligned}$$

with $A_n = (1-aq^{n+1})(1-bq^{n+1})$ and $C_n = -abq^{n+1}(1-q^n)$. If we take $b = q^{-N}$, with the help of table 2, we see that $\mathscr{L}_0$ is the linear operator known for the affine q-Krawtchouk $k_n^{aff}(x;a,N-1;q)$, i.e.,

$$\mathscr{L}_0(p) = \sum_{x=0}^{N-1} \frac{(aq;q)_x(q;q)_{N-1}}{(q;q)_x(q;q)_{N-1-x}}\ (aq)^{-x} p\left(q^{-x}\right)$$

The operator $\mathscr{T}$ is $\mathscr{D}_q$, since

$$\mathscr{D}_q p_n(x;a,q^{-N};q) = const.p_{n-1}(qx;aq,q^{-N+1};q),$$

which shifts down one position the vanishing coefficient γ_n of the new TTRR. Thus, after applying the linear operator $\mathscr{D}_q$ N times one obtains the big q-Laguerre polynomials $p_{n-N}(x;aq^N,1;q)$ for which $\Lambda = \emptyset$ and

$$\mathscr{L}_N(p) = \int_q^{aq^{N+1}} p(x)\ (a^{-1}q^{-N}x;q)_\infty d_qx.$$

5.3. q-Meixner

The monic q-Meixner polynomials, $m_n(x;b,c;q)$, satisfy the TTRR

$$\begin{aligned} xm_n(x) = m_{n+1}(x) + \big(1 + q^{-2n-1}(c(1-bq^{n+1}) + q(1-q^n)(c+q^n))\big)m_n(x) \\ - cq^{-4n+1}(1-q^n)(1-bq^n)(c+q^n)m_{n-1}(x). \end{aligned}$$

If we take $b = q^{-N}$, with the help of table 2, we see that $\mathscr{L}_0$ is the linear operator known for the quantum q-Krawtchouk $k_n^{qtm}(x;-c^{-1},N-1;q)$, i.e.,

$$\mathscr{L}_0(p) = \sum_{x=0}^{N-1} \frac{(-c^{-1}q;q)_{N-1-x}}{(q;q)_x(q;q)_{N-1-x}}\ (-1)^{N-1-x} q^{\binom{x}{2}} p\left(q^{-x}\right)\ .$$

The operator $\mathscr{T}$ is $\mathscr{D}_{q^{-1}}$, since

$$\mathscr{D}_{q^{-1}} m_n(x;q^{-N},c;q) = const.m_{n-1}(x;q^{-N+1},cq^{-1};q),$$

which shifts down one position the vanishing coefficient γ_n of the new TTRR. Thus, after applying $\mathscr{D}_{q^{-1}}$ N times one obtains the q-Meixner polynomials $p_{n-N}(x;1,cq^{-N};q)$ for which $\Lambda = \emptyset$ and

$$\mathscr{L}_N(p) = \sum_{x=0}^{\infty} \frac{(cq^{-N})^x\ q^{\binom{x}{2}}}{(-cq^{-N+1};q)_x}\ p(q^{-x}).$$

The monic little q-Jacobi polynomials, $p_n(x;a,b|q)$, satisfy the TTRR

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + (A_n + C_n)p_n(x) \\ &\quad + aq^{2n-1}\frac{(1-q^n)(1-aq^n)(1-abq^n)(1-bq^n)}{(1-abq^{2n-1})(1-abq^{2n})(1-abq^{2n+1})}p_{n-1}(x)\,. \end{aligned}$$

If we take $b = q^{-N}$, with the help of table 2, we see that $\mathscr{L}_0$ is the linear operator known for the q-Krawtchouk $k_n(q^{-N+1}x; -aq^{-N+1}, N-1; q)$, i.e.,

$$\mathscr{L}_0(p) = \sum_{x=0}^{N-1} \frac{(q^{x-N+1};q)_x}{(q;q)_x}\,(aq^{-N+1})^{-x}\; p\left(q^{N-1-x}\right)\,.$$

The operator $\mathscr{T}$ is $\mathscr{D}_q$, since

$$\mathscr{D}_q p_n(x;a,q^{-N}|q) = const. p_{n-1}(x;aq,q^{-N+2}|q),$$

which shifts down one position the vanishing coefficient γ_n of the new TTRR. Thus, after applying $\mathscr{D}_q$ N times one obtains the little q-Jacobi polynomials $p_{n-N}(x;aq^N,1|q)$ for which $\Lambda = \emptyset$ and

$$\mathscr{L}_N(p) = \sum_{k=0}^{\infty}(aq^{N+1})^k\; p(q^k).$$

Family and condition		Nth associated polynomial	
AW	$b = a^{-1}q^{-N+1}$	AW	$p_n(x;aq^N,a^{-1}q,c,d;q)$
bqJ	$c = q^{-N}$	bqJ	$p_n(xq^N;aq^N,bq^N,q^N;q)$
cd$q$H	$c = a^{-1}q^{-N+1}$	cd$q$H	$p_n(x;aq^N,b,a^{-1}q;q)$
bqL	$b = q^{-N}$	bqL	$p_n(xq^N;aq^N,q^N;q)$
qM	$b = q^{-N}$	qM	$m_n(xq^N;q^N,cq^{-N};q)$
lqJ	$b = q^{-N}$	lqJ	$p_n(xq^{-N};a,q^N;q)$

Table 1: Nth associated polynomials involved in the factorization

Acknowledgment

We are grateful for the exhaustive work done by the referees. Their comments and suggestions have improved the presentation of the manuscript. We also thank to J. J. Moreno-Balcázar for his useful comments.

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qR→ AW	$r_n(x;\alpha,\beta,\gamma,\delta;q)$	$p_n\left(\frac{x}{2\sqrt{\gamma\delta q}};\sqrt{\gamma\delta q},\ \alpha\sqrt{\frac{q}{\gamma\delta}},\ \beta\sqrt{\frac{\delta q}{\gamma}},\ \sqrt{\frac{\gamma q}{\delta}};q\right)$
AW→ qR	$p_n(x;a,b,c,d;q)$	$r_n\left(2ax;\frac{ab}{q},\frac{cd}{q},\frac{ad}{q},\frac{a}{d};q\right)$
bqJ→ qH	$p_n(x;a,b,c;q)$	$h_n(x;a,b,-1-\log_q c\,;q)$
qH→bqJ	$h_n(x;a,b,N;q)$	$p_n(x;a,b,q^{-N-1};q)$
dqH→cd$q$H	$r_n(x;\gamma,\delta,N;q)$	$p_n\left(\frac{x}{2\sqrt{\gamma\delta q}};\sqrt{\gamma\delta q},\ \sqrt{\frac{\gamma q}{\delta}},\ \frac{1}{q^N\sqrt{\gamma\delta q}}\vert q\right)$
cd$q$H→dqH	$p_n(x;a,b,c\vert q)$	$r_n\left(2ax;\frac{ab}{q},\frac{a}{b},-\log_q(ac);q\right)$
QqK→ qM	$k_n^{qtm}(x;p,N;q)$	$m_n\left(x;q^{-N-1},-\frac{1}{p};q\right)$
qM→QqK	$m_n(x;b,c;q)$	$k_n^{qtm}\left(x;-1-\log_q b,-\frac{1}{c};q\right)$
QqK→AqK	$k_n^{qtm}(x;p,N;q)$	$k_n^{aff}\left(xq^N;p^{-1},N;q^{-1}\right)$
AqK→QqK	$k_n^{aff}(x;p,N;q)$	$k_n^{qtm}(xq^{-N};p^{-1},N;q^{-1})$
qK→lqJ	$k_n(x;p,N;q)$	$p_n(xq^N;-pq^N,q^{-N-1};q)$
lqJ→ qK	$p_n(x;a,b;q)$	$k_n(bqx;-abq,-1-\log_q b;q)$
AqK→bqL	$k_n^{aff}(x;p,N;q)$	$p_n(x;p,q^{-N-1};q)$
bqL→AqK	$p_n(x;a,b;q)$	$k_n^{aff}(x;a,-1-\log_q b;q)$
lqJ→bqJ	$p_n(x;a,b;q)$	$p_n(bqx;b,a,0;q)$
qK→ bqJ	$k_n(x;p,N;q)$	$p_n(x;q^{-N-1},-pq^N,0;q)$

Table 2: Some unnormalized identities between q-polynomials.

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