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Multi-integral representations for associated Legendre and Ferrers functions

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Abstract: For the associated Legendre and Ferrers functions of the first and second kind, we obtain new multi-derivative and multi-integral representation formulas. The multi-integral representation formulas that we derive for these functions generalize some classical multi-integration formulas. As a result of the determination of these formulae, we compute some interesting special values and integral representations for certain particular combinations of the degree and order including the case where there is symmetry and antisymmetry for the degree and order parameters. As a consequence of our analysis, we obtain some new results for the associated Legendre function of the second kind including parameter values for which this function is identically zero.

Keywords: Associated Legendre functions; Ferrers functions; Integral representations; Gauss hypergeometric function

1. Introduction

Using analysis for fundamental solutions of the Laplace equation on Riemannian manifolds of constant curvature, we have previously obtained antiderivatives and integral representations for the associated Legendre and Ferrers functions of the second kind with degree and order equal to within a sign. For instance, in [1, Theorem 1], using the d-dimensional hypersphere with $d = 2, 3, 4, \ldots$, we derive an antiderivative and an integral representation for the Ferrers function of the second kind with order equal to the negative degree. In [2, Theorem 3.1], using the d-dimensional hyperboloid model of hyperbolic geometry with $d = 2, 3, 4, \ldots$, the authors derived an antiderivative and an integral representation for the associated Legendre function of the second kind with degree and order equal to each other. In [1,2], the antiderivatives and integral representations were restricted to values of the degree and order ν such that 2ν is an integer. One of the goals of this paper is to generalize some integral representation results presented in [1,2] for the associated Legendre and Ferrers functions of the first and second kind, and to extend them such that the degree and order are no longer subject to the above restriction. Our integral representations are consistent with known special values for the associated Legendre and Ferrers functions of the first kind when the order is equal to the negative degree.

The multi-integrals presented in this paper (Theorems 3.10, 3.17, 4.3, 4.6, 4.11 and 4.11) generalize multi-integrals for arbitrary order (μ), which have appeared previously in the literature (see for instance, [3, Section 14.6(ii)], [4, (8.14.17-18)] and the earliest appearance we have found [5, p. 149]). The multi-integrals for the Ferrers function of the second kind (Theorems 4.16, 4.20, 4.26 and 4.30) also produce generalized results for arbitrary order (μ). In fact, the specialization of these multi-integrals for $\mu = 0$ has not appeared in the literature.

Applications of the work contained in this manuscript include any of the many different areas in which Legendre and Ferrers functions arise, which include a very large number of disciplines. The associated Legendre and Ferrers functions treated in this paper, e.g., $Q_\nu^\mu(\cosh r)/\sinh^\mu r$, appear as fundamental solutions of the Laplace and Helmholtz equation on Riemannian manifolds of constant curvature (see e.g., [6, Section 3.3]). Associate Legendre and Ferrers functions appear in any place where harmonic analysis needs to be performed on the surface of a sphere or on an oblate or prolate spheroid. These are analogs of the $1/r$ potential in Euclidean space for Riemannian spaces of constant curvature. Therefore results such as we derive below will be important when studying global analysis of these fundamental solutions for higher powers of the Laplacian or Helmholtz operators. These multi-integration results provide an algorithm for computing fundamental solutions of much larger powers of the Laplace-Beltrami operator on these spaces. A survey of applications of associated Legendre and Ferrers functions is given in [3, Sections 14.30-31]. This points to harmonic analysis on the surface of spheres, oblate and prolate spheroids, and circular toroids. Other applications include the Mehler-Fock transforms, high frequency atomic and molecular scattering, quantum direct and exchange Coulomb interaction, Newtonian gravity, etc. Also, the special cases treated in this paper, many of which are also not well-known, provide for a beautiful illumination of this classical subject.

The critical aspects that allows for proofs of the results contained in this paper is that fact that we are able to obtain differentiation/integration properties of associated Legendre and Ferrers functions such that the order (μ) of these functions are either raised or lowered by integral amounts. Furthermore, the degree (ν) of these functions are not at all affected by the differentiation/integration. In fact, these differentiation/integration properties (Remarks 3.9, 3.15, 4.1, 4.7, 4.15, 4.19, 4.24 and 4.28) are not well-known in the literature for these functions.

Note that in the process of our derivations, we also obtained some nice results for the associated Legendre function of the second kind with degree $\nu = -\frac{3}{2} - n$. This included the full Gauss hypergeometric dependence and large argument asymptotics which is crucial for establishing Theorem 3.10. We are also able to find their zeros which occur when $\mu = \pm(\frac{1}{2} + k)$, $n, k \in \mathbb{N}_0$, $n \geq k$ (see Corollary 3.4).

2. Preliminaries

Throughout this paper we adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and we use the set $\mathbb{C}$ which represents the complex numbers. As is the common convention for associated Legendre functions [4, (8.1.1)], for any expression of the form $(z^2-1)^\alpha$, read this as $(z^2-1)^\alpha := (z+1)^\alpha (z-1)^\alpha$, for any fixed $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 1]$. In this paper, we will use the Gauss hypergeometric function ${}_2F_1$ which can be defined in terms of the following infinite series as [7, (2.1.5)]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $c \notin -\mathbb{N}_0$, and elsewhere on $z \in \mathbb{C} \setminus (1, \infty)$ by analytic continuation; where the Pochhammer symbol (rising factorial) is defined by

$$(z)_n := \prod_{i=1}^{n} (z + i - 1), \tag{2.1}$$

with $n \in \mathbb{N}_0$. Note that for all $n \in \mathbb{N}_0$, $z \notin -\mathbb{N}_0$, one has

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad \Gamma(z-n) = \frac{(-1)^n \Gamma(z)}{(-z+1)_n}. \tag{2.2}$$

We will also need the binomial theorem [3, (15.4.6)]

$$ {}_2F_1\left(\begin{matrix}a,b\\ b\end{matrix};z\right)=(1-z)^{-a}, \tag{2.3}$$

Euler's transformation (2.4), [3, (15.8.1)]

$$ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\ c\end{matrix};z\right), \tag{2.4}$$

and the Gauss sum [3, (15.4.20)]

$$ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.5}$$

for $\Re(c-a-b)>0$. We will also use the generalized hypergeometric function

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};z\right):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{z^n}{n!},$$

where $d,e\notin-\mathbb{N}_0$. We now produce a lemma for the Gauss hypergeometric function which will be useful in our analysis of antiderivatives for associated Legendre functions of the second kind below.

Lemma 2.1. *Let $z\in\mathbb{C}\setminus[0,\infty)$. Then*

$$ \frac{d}{dz}\frac{1}{z^{\nu+\mu+1}}\,{}_2F_1\left(\begin{matrix}\frac{\nu+\mu+1}{2},\frac{\nu+\mu+2}{2}\\ \nu+\frac{3}{2}\end{matrix};\frac{1}{z^2}\right)=\frac{-(\nu+\mu+1)}{z^{\nu+\mu+2}}\,{}_2F_1\left(\begin{matrix}\frac{\nu+\mu+2}{2},\frac{\nu+\mu+3}{2}\\ \nu+\frac{3}{2}\end{matrix};\frac{1}{z^2}\right). \tag{2.6}$$

Proof. Differentiating the left-hand side of (2.6) using the chain rule and [3, (15.5.1)]

$$ \frac{d}{dz}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=\frac{ab}{c}\,{}_2F_1\left(\begin{matrix}a+1,b+1\\ c+1\end{matrix};z\right), \tag{2.7}$$

one produces an expression involving the sum of two Gauss hypergeometric functions. One can then use the following Gauss relations for contiguous hypergeometric functions [5, p. 58]

$$ z\,{}_2F_1\left(\begin{matrix}a+1,b+1\\ c+1\end{matrix};z\right)=\frac{c}{a-b}\left[{}_2F_1\left(\begin{matrix}a,b+1\\ c\end{matrix};z\right)-{}_2F_1\left(\begin{matrix}a+1,b\\ c\end{matrix};z\right)\right], \tag{2.8}$$

and [3, (15.5.12)]

$$ {}_2F_1\left(\begin{matrix}a,b+1\\ c\end{matrix};z\right)=\frac{b-a}{b}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)+\frac{a}{b}\,{}_2F_1\left(\begin{matrix}a+1,b\\ c\end{matrix};z\right), \tag{2.9}$$

to obtain the following formula

$$ \frac{d}{dz}\frac{1}{z^{a+b-\frac{1}{2}}}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};\frac{1}{z^2}\right)=\frac{a-b+\frac{1}{2}}{z^{a+b+\frac{1}{2}}}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};\frac{1}{z^2}\right)-\frac{2a}{z^{a+b+\frac{1}{2}}}\,{}_2F_1\left(\begin{matrix}a+1,b\\ c\end{matrix};\frac{1}{z^2}\right). \tag{2.10}$$

Since for the Gauss hypergeometric function on the left-hand side of (2.6), $a-b+\frac{1}{2}=0$, so the first term on the right-hand side of (2.10) vanishes and the lemma follows. □

Definition 2.2. *Let $z \in \mathbb{C}$, $a, b \in \mathbb{C} \cup \{-\infty, \infty\}$. Define the following notations for nth iterated integrals of the functions $f(z;\mathbf{a})$, $g(z;\mathbf{b})$, respectively,*

$$\int_z^b \cdots \int_z^b f(w;\mathbf{a})(dw)^n := \int_z^b \left[\int_{w_{n-1}}^b \cdots \left[\int_{w_2}^b \left[\int_{w_1}^b f(w;\mathbf{a})\,dw\right] dw_1\right] \cdots dw_{n-2}\right] dw_{n-1}, \tag{2.11}$$

$$\int_a^z \cdots \int_a^z g(w;\mathbf{b})(dw)^n := \int_a^z \left[\int_a^{w_{n-1}} \cdots \left[\int_a^{w_2} \left[\int_a^{w_1} g(w;\mathbf{b})\,dw\right] dw_1\right] \cdots dw_{n-2}\right] dw_{n-1}, \tag{2.12}$$

where $w_0 := w$, $w_n := z$, and $\mathbf{a}$, $\mathbf{b}$, are sets of fixed parameters.

Another useful result we are going to use often along this work is the following.

Lemma 2.3. *Let $n \in \mathbb{N}_0$, $a, x, \mu \in \mathbb{C}$, and let f^μ be a function such that*

$$\frac{d}{dz} f^\mu(z) = \lambda_\mu f^{\mu\pm 1}(z), \tag{2.13}$$

where $\lambda_\mu \in \mathbb{C}^*$. Then, the following identity holds:

$$\int_a^x \cdots \int_a^x f^\mu(w)(dw)^n = \frac{1}{\lambda_{\mu\mp 1} \cdots \lambda_{\mu\mp n}} \sum_{k=n}^{\infty} \frac{\lambda_{\mu\mp n} \cdots \lambda_{\mu\mp n\pm(k-1)} f^{\mu\mp n\pm k}(a)(x-a)^k}{k!}.$$

Proof. We are going to prove the result by induction on n. The $n = 0$ case is direct taking into account the Taylor expansion of f at $x = a$. If $n = 1$ then

$$\int_a^x f^\mu(w)dw = \frac{1}{\lambda_{\mu\mp 1}} \int_a^x \left(\frac{d}{dw} f^{\mu\mp 1}(w)\right) dw = \frac{1}{\lambda_{\mu\mp 1}} \left(f^{\mu\mp 1}(x) - f^{\mu\mp 1}(a)\right).$$

By using the Taylor expansion of $f^{\mu\mp 1}(x)$ at $x = a$ and (2.13), the result follows for the $n = 1$ case. Assuming the result holds for n, let us prove the identity for the $n + 1$ case:

$$\begin{aligned}\int_a^x \cdots \int_a^x f^\mu(w)(dw)^n &= \frac{1}{\lambda_{\mu\mp 1}} \int_a^x \cdots \int_a^x \left(f^{\mu\mp 1}(w_1) - f^{\mu\mp 1}(a)\right) dw_1 \cdots dw_n \\ &= \frac{1}{\lambda_{\mu\mp 1} \cdots \lambda_{\mu\mp(1+n)}} \sum_{k=n+1}^{\infty} \frac{\lambda_{\mu\mp(n+1)} \cdots \lambda_{\mu\mp(n+1)\pm(k-1)} f^{\mu\mp(n+1)\pm k}(a)(x-a)^k}{k!},\end{aligned}$$

where we have used induction and the basic properties of integrals. Hence the result follows. □

3. Associated Legendre functions of the first and second kind

Associated Legendre functions (and Ferrers functions) are those Gauss hypergeometric functions which satisfy a quadratic transformation (see [3, Sections 15.8(iii-iv)]). In the following sections we will derive derivative, antiderivative, and integral representations for associated Legendre (and Ferrers) functions of the first and second kinds which to the best of our knowledge have not appeared in the classical literature of these highly applicable special functions of applied and pure mathematics.

The associated Legendre function of the first kind $P_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}$ is defined as [3, (14.3.6)]

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{matrix}-\nu, \nu+1 \\ 1-\mu\end{matrix}; \frac{1-z}{2}\right). \tag{3.1}$$

Starting with (3.1), setting $\mu \mapsto -\mu$, and applying (2.4), another useful hypergeometric representation for the associated Legendre function of the first kind can be obtained, namely

$$P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\frac{\mu}{2}}}{2^\mu \Gamma(\mu+1)} \, {}_2F_1\left(\begin{matrix} \nu+\mu+1, -\nu+\mu \\ 1+\mu \end{matrix} ; \frac{1-z}{2} \right). \tag{3.2}$$

The associated Legendre function of the second kind $\boldsymbol{Q}_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}$ can be defined in terms of the Gauss hypergeometric function as [3, (14.3.10) and Section 14.21]

$$\boldsymbol{Q}_\nu^\mu(z) := \frac{\sqrt{\pi}(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}} \, {}_2F_1\left(\begin{matrix} \frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2} \\ \nu+\frac{3}{2} \end{matrix} ; \frac{1}{z^2} \right), \tag{3.3}$$

for $|z| > 1$ and, by analytic continuation of the Gauss hypergeometric function, elsewhere on $z \in \mathbb{C} \setminus (-\infty, 1]$.

Remark 3.1. *The normalized notation* $\boldsymbol{Q}_\nu^\mu(z)$ *is due to Olver [8, p. 178] and is defined in terms of the more commonly appearing Hobson notation for the associated Legendre function of the second kind* $Q_\nu^\mu(z)$ *as follows [3, (14.3.10)]*

$$\boldsymbol{Q}_\nu^\mu(z) = \frac{\mathrm{e}^{-i\pi\mu}}{\Gamma(\nu+\mu+1)} Q_\nu^\mu(z). \tag{3.4}$$

See [3, Section 14.1] for more information on commonly appearing notations for the associated Legendre and Ferrers functions.

Remark 3.2. *Note that the following algebraic special cases for the associated Legendre function of the second kind, hold for* $\mu = \nu+1, \nu+2$, *namely*

$$\boldsymbol{Q}_\nu^{\nu+1}(z) = \frac{\sqrt{z^2-1}}{z} \boldsymbol{Q}_\nu^{\nu+2}(z) = \frac{\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})(z^2-1)^{\frac{\nu+1}{2}}}, \tag{3.5}$$

where we have used (3.3) *and* (2.3). *Furthermore algebraic expressions for* $\boldsymbol{Q}_\nu^{\nu+n}$ *for all* $n \in \mathbb{N}$ *are obtainable from the order recurrence relation for associated Legendre functions of the second kind (cf. [3, (14.10.6)])*

$$\boldsymbol{Q}_\nu^{\nu+n+2}(z) = \frac{2(\nu+n+1)z}{(2\nu+n+2)\sqrt{z^2-1}} \boldsymbol{Q}_\nu^{\nu+n+1}(z) - \frac{n}{2\nu+n+2} \boldsymbol{Q}_\nu^{\nu+n}(z). \tag{3.6}$$

Since $\boldsymbol{Q}_\nu^{\nu+2}(z)$ *is proportional to* $\boldsymbol{Q}_\nu^{\nu+1}(z)$, *then all* $\boldsymbol{Q}_\nu^{\nu+m}(z)$ *is proportional to* $\boldsymbol{Q}_\nu^{\nu+1}(z)$ *for all* $m \geq 2$.

We now present some theorems related to the behavior of the associated Legendre function of the second kind with degree $\nu = -\frac{3}{2} - n \in \{-\frac{3}{2}, -\frac{5}{2}, \ldots\}$, $n \in \mathbb{N}_0$ and its corresponding asymptotics as $z \to \infty$. This will be useful in our further analysis below.

Theorem 3.3. *Let* $z \in \mathbb{C} \setminus (-\infty, 1]$, $\mu \in \mathbb{C}$, $\nu = -\frac{3}{2} - n \in \{-\frac{3}{2}, -\frac{5}{2}, \ldots\}$, $n \in \mathbb{N}_0$. *Then*

$$\boldsymbol{Q}^\mu_{-\frac{3}{2}-n}(z) = \frac{(-1)^n \sqrt{\pi}(\mu^2-\frac{1}{4})(\frac{3}{2}+\mu)_n(\frac{3}{2}-\mu)_n(z^2-1)^{\frac{\mu}{2}}}{2^{n+\frac{3}{2}}(n+1)!\, z^{n+\frac{3}{2}+\mu}} \, {}_2F_1\left(\begin{matrix} \frac{3}{4}+\frac{\mu+n}{2}, \frac{5}{4}+\frac{\mu+n}{2} \\ n+2 \end{matrix} ; \frac{1}{z^2} \right). \tag{3.7}$$

Proof. Start with (3.3) then let $\nu = -\frac{3}{2} - n$, followed by the application of [3, First equation in Section 15.2(ii)],

$$\lim_{c \to -n} \frac{1}{\Gamma(c)} \, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \frac{(a)_{n+1}(b)_{n+1}}{(n+1)!} z^{n+1} \, {}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ n+2 \end{matrix} ; z \right),$$

and the Pochhammer symbol identity for $\nu \in \mathbb{C}$, $n \in \mathbb{N}_0$,

$$\left(\nu - \frac{n}{2}\right)_n \left(\nu + \frac{1}{2} - \frac{n}{2}\right)_n = \frac{(-1)^n}{2^{2n}} (2\nu)_n (-2\nu + 1)_n,$$

which follows from the duplication theorem for gamma functions then (2.2). □

The following corollary is an interesting side-effect of the above theorem which produces zeros for the associated Legendre function of the second kind.

Corollary 3.4. *Let* $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu = -\frac{3}{2} - n \in \{-\frac{3}{2}, -\frac{5}{2}, \ldots\}$, $\mu = \pm(\frac{1}{2} + k)$, $n, k \in \mathbb{N}_0$. *Then*

$$\boldsymbol{Q}_\nu^\mu(z) = \boldsymbol{Q}_{-\frac{3}{2}-n}^{\pm(\frac{1}{2}+k)}(z) = 0, \tag{3.8}$$

for all $n \geq k$.

Proof. Simple examination of the factor multiplying the Gauss hypergeometric function in (3.7) produces the result. □

Remark 3.5. *Note that the zeros for associated Legendre functions for the* $k = 0$ *case in Corollary 3.4 is clear from the special value [3, (14.5.17)]*

$$\boldsymbol{Q}_\nu^{\pm\frac{1}{2}}(\cosh \xi) = \sqrt{\frac{\pi}{2 \sinh \xi}} \frac{\exp(-(\nu + \frac{1}{2})\xi)}{\Gamma(\nu + \frac{3}{2})}.$$

We now give a result which produces the large argument asymptotics for the associated Legendre function of the second kind when the degree $\nu = -\frac{3}{2} - n$, $n \in \mathbb{N}_0$.

Lemma 3.6. *Let* $z \in \mathbb{C} \setminus (-\infty, 1]$, $\mu \in \mathbb{C}$, $\nu = -\frac{3}{2} - n \in \{-\frac{3}{2}, -\frac{5}{2}, \ldots\}$, $n \in \mathbb{N}_0$. *Then*

$$\boldsymbol{Q}_\nu^\mu(z) \sim \frac{(-1)^{-\nu-\frac{3}{2}} \sqrt{\pi}\, 2^\nu (\mu^2 - \frac{1}{4}) \Gamma(\mu - \nu) \Gamma(-\mu - \nu) z^\nu}{\Gamma(\frac{1}{2} - \nu) \Gamma(\frac{3}{2} + \mu) \Gamma(\frac{3}{2} - \mu)}. \tag{3.9}$$

Equivalently,

$$\boldsymbol{Q}_{-\frac{3}{2}-n}^\mu(z) \sim \frac{(-1)^n \sqrt{\pi}\, (\mu^2 - \frac{1}{4}) (\frac{3}{2} + \mu)_n (\frac{3}{2} - \mu)_n}{2^{n+\frac{3}{2}} (n+1)! z^{n+\frac{3}{2}}}. \tag{3.10}$$

Proof. The result follows by starting with (3.7) and examining its leading term behavior as $z \to \infty$. □

3.1. The associated Legendre function of the second kind

We now compute some antiderivatives and integral representations for associated Legendre functions of the second kind. This also includes some nice limits and specializations.

Remark 3.7. *Note the following expression can be obtained by using the definition (3.3) and Lemma 2.1 for $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu, \mu \in \mathbb{C}$:*

$$\frac{d}{dz}\frac{\boldsymbol{Q}_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}} = \frac{-(\nu+\mu+1)}{(z^2-1)^{\frac{\mu+1}{2}}}\boldsymbol{Q}_\nu^{\mu+1}(z). \tag{3.11}$$

From this formula the following antiderivative is obtained:

$$\int \frac{\boldsymbol{Q}_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}}dz = \frac{-\boldsymbol{Q}_\nu^{\mu-1}(z)}{(\nu+\mu)(z^2-1)^{\frac{\mu-1}{2}}} + C, \tag{3.12}$$

where C is an arbitrary constant.

Theorem 3.8. *Let $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu, \mu \in \mathbb{C}$, such that $\Re(\nu+\mu+1) > 0$. Then*

$$\boldsymbol{Q}_\nu^\mu(z) = (\nu+\mu+1)(z^2-1)^{\frac{\mu}{2}} \int_z^\infty \frac{\boldsymbol{Q}_\nu^{\mu+1}(w)}{(w^2-1)^{\frac{\mu+1}{2}}}dw. \tag{3.13}$$

Proof. Taking the limit of the antiderivative (3.12) evaluated at the endpoints of integration using the large argument asymptotics [3, (14.8.15)]

$$\boldsymbol{Q}_\nu^\mu(z) \sim \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{3}{2})(2z)^{\nu+1}}, \quad \nu \notin \{-\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{7}{2}, \ldots\},$$

and Lemma 3.6 which shows that $\boldsymbol{Q}_\nu^\mu(z) \to 0$ as $z \to \infty$ for $\nu \in \{-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots\}$ as well. Therefore the integral is convergent as indicated which completes the proof. ☐

Remark 3.9. *Iterating (3.11), then using induction with (2.1), the following order-shift derivative formula for the associated Legendre function of the second kind, namely for $z \in \mathbb{C} \setminus (-\infty, 1]$, $n \in \mathbb{N}_0$, $\nu, \mu \in \mathbb{C}$, holds:*

$$\frac{d^n}{dz^n}\frac{\boldsymbol{Q}_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}} = \frac{(-1)^n(\nu+\mu+1)_n}{(z^2-1)^{\frac{\mu+n}{2}}}\boldsymbol{Q}_\nu^{\mu+n}(z). \tag{3.14}$$

Theorem 3.10. *Let $n \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $n \in \mathbb{N}_0$, $\nu, \mu \in \mathbb{C}$, such that $\Re(\nu+\mu-n+1) > 0$. Then*

$$\int_z^\infty \cdots \int_z^\infty \frac{\boldsymbol{Q}_\nu^\mu(w)}{(w^2-1)^{\frac{\mu}{2}}}(dw)^n = \frac{(-1)^n\boldsymbol{Q}_\nu^{\mu-n}(z)}{(-\nu-\mu)_n(z^2-1)^{\frac{\mu-n}{2}}}. \tag{3.15}$$

Proof. Iterating Theorem 3.8 with (2.1) using induction with (2.1) completes the proof. ☐

Remark 3.11. *It is clear that Theorem 3.10 is a generalization of cf. [3, (14.6.8)]*

$$\boldsymbol{Q}_\nu^{-n}(z) = (-1)^n(-\nu)_n(z^2-1)^{-\frac{n}{2}}\int_z^\infty \cdots \int_z^\infty \boldsymbol{Q}_\nu(w)(dw)^n, \tag{3.16}$$

by considering the $\mu = 0$ specialization in (3.15), using (2.2) and Hobson's notation (see Remark 3.1).

Remark 3.12. *An antiderivative of an algebraic function (essentially in terms of reciprocal powers of the hyperbolic sine function) expressed as the associated Legendre function of the second kind with order and degree equal to each other can be obtained . This is accomplished by starting with (3.12) and setting $\mu =$*

$\nu+1$, then using (3.5). This produces the specialized antiderivative, namely for $z \in \mathbb{C}\setminus(-\infty,1]$, $\nu \in \mathbb{C}\setminus\{-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots\}$,

$$\int \frac{dz}{(z^2-1)^{\nu+1}} = \frac{-1}{(2\nu+1)z^{2\nu+1}}\,{}_2F_1\left(\begin{array}{c}\nu+\frac{1}{2},\nu+1\\ \nu+\frac{3}{2}\end{array};\frac{1}{z^2}\right)+C=\frac{-2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(z^2-1)^{\frac{\nu}{2}}}\boldsymbol{Q}_{\nu}^{\nu}(z)+C,$$

where C is an arbitrary constant.

A straightforward consequence of the antiderivative (3.17) is the following integral representation for the associated Legendre function of the second kind with degree and order equal to each other.

Corollary 3.13. *Let $\nu \in \mathbb{C}$ such that $\Re\nu > -\frac{1}{2}$, $z \in \mathbb{C}\setminus(-\infty,1]$. Then*

$$\boldsymbol{Q}_{\nu}^{\nu}(z) = \boldsymbol{Q}_{\nu}^{-\nu}(z) = \frac{\sqrt{\pi}(z^2-1)^{\frac{\nu}{2}}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}\int_z^{\infty}\frac{dw}{(w^2-1)^{\nu+1}}. \tag{3.17}$$

Proof. Evaluating the antiderivative Theorem 3.12 at the endpoints of integration and taking advantage of [3, (14.9.14)]

$$\boldsymbol{Q}_{\nu}^{-\mu}(z) = \boldsymbol{Q}_{\nu}^{\mu}(z),$$

completes the proof. □

3.2. *The associated Legendre function of the first kind*

An integral representation for the associated Legendre function of the first kind by applying the Whipple formulae to (3.14) can be obtained . However, this integral representation shifts the degree of the associated Legendre function of the first kind ν by unity instead of shifting the order by unity.

Corollary 3.14. *Let $z \in \mathbb{C}\setminus(-\infty,1]$, $\nu,\mu \in \mathbb{C}$. Then*

$$P_{\nu-1}^{-\mu}(z) = (\nu+\mu)(z^2-1)^{-\frac{\nu}{2}}\int_1^{z}\frac{P_{\nu}^{-\mu}(w)}{(w^2-1)^{\frac{\nu+2}{2}}}dw. \tag{3.18}$$

Proof. Apply the Whipple formula [3, (14.9.16)]

$$\boldsymbol{Q}_{\nu}^{\mu}(z) = \sqrt{\frac{\pi}{2}}(z^2-1)^{-1/4}P_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right), \tag{3.19}$$

to the associated Legendre functions of the second kind on the left and right-hand sides of (3.14) followed by the application of the involution [9, Section 2] $\zeta(z) := \log\coth\frac{z}{2}$ and making a change of variables $w = \zeta/\sqrt{\zeta^2-1}$ completes the proof. □

The following integral representation can be derived by applying Whipple's formulae to our integral representation for the associated Legendre function of the second kind. We are able to obtain an integral representation for the associated Legendre function of the first kind which shifts the order by an integer value, similar to (3.15). This is achieved by deriving a corresponding derivative formula as follows.

Remark 3.15. *If you divide both sides of (3.2) by $(z^2-1)^{\frac{\mu}{2}}$ and differentiate with respect to z, by using (2.7), the unit increments of the parameters of the Gauss hypergeometric function can be absorbed in the order (μ) of the associated Legendre function of the first kind. Let $n \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu, \mu \in \mathbb{C}$. Then*

$$\frac{d^n}{dz^n}\frac{P_\nu^{-\mu}(z)}{(z^2-1)^{\frac{\mu}{2}}} = \frac{(-1)^n(\nu+\mu+1)_n(\mu-\nu)_n}{(z^2-1)^{\frac{\mu+n}{2}}}P_\nu^{-\mu-n}(z). \tag{3.20}$$

From the above result integral representations can be obtained through repeated integration. For instance, the single integral result is given as follows.

Corollary 3.16. *Let $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu, \mu \in \mathbb{C}$. Then*

$$\int_1^z \frac{P_\nu^{-\mu}(w)}{(w^2-1)^{\frac{\mu}{2}}}dw = \frac{1}{(\nu+\mu)(\nu-\mu+1)}\left(\frac{P_\nu^{-\mu+1}(z)}{(z^2-1)^{\frac{\mu-1}{2}}} - \frac{1}{2^{\mu-1}\Gamma(\mu)}\right). \tag{3.21}$$

Proof. In order to derive this result, after applying the fundamental theorem of calculus for some continuous function f on $[a,b]$,

$$\int_a^b f'(t)dt = f(b) - f(a), \tag{3.22}$$

and taking advantage of [3, (14.8.7)]

$$\lim_{z\to 1^+}\frac{P_\nu^{-\mu}(z)}{(z^2-1)^{\frac{\mu}{2}}} = \frac{1}{2^\mu\Gamma(\mu+1)}, \tag{3.23}$$

this completes the proof. ☐

The above result can be generalized by repeatedly integrating the above formula.

Theorem 3.17. *Let $n \in \mathbb{N}_0$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\nu, \mu \in \mathbb{C}$. Then*

$$\begin{aligned}\int_1^z \cdots \int_1^z \frac{P_\nu^{-\mu}(w)}{(w^2-1)^{\frac{\mu}{2}}}(dw)^n &= \frac{1}{(-\nu-\mu)_n(\nu-\mu+1)_n}\left(\frac{(-1)^n P_\nu^{-\mu+n}(z)}{(z^2-1)^{\frac{\mu-n}{2}}}\right.\\ &\quad\left. - \frac{(-\mu)_n}{2^{\mu-n}\Gamma(\mu+1)}\sum_{k=0}^{n-1}\frac{(\nu+\mu+1-n)_k(\mu-\nu-n)_k}{k!(\mu+1-n)_k}\left(\frac{1-z}{2}\right)^k\right) &(3.24)\\ &= \frac{(z-1)^n}{2^\mu n!\Gamma(\mu+1)}\,{}_3F_2\left(\begin{matrix}\nu+\mu+1, \mu-\nu, 1\\ \mu+1, n+1\end{matrix}; \frac{1-z}{2}\right). &(3.25)\end{aligned}$$

Proof. Repeated integration of (3.20) while noting (3.23), and using induction with (2.1) derives the two sum expression (4.9). By rewriting the associated Legendre function of the first kind on the right-hand side of (3.25) in terms of the Gauss hypergeometric representation (3.2), the finite sum term cancels the first n terms of the k sum, and rewriting the resulting expression shows it can be written in terms of a nonterminating ${}_3F_2$. This completes the proof. ☐

Remark 3.18. *It is clear that Theorem 3.17 is a generalization of [3, (14.6.7)]*

$$P_\nu^{-n}(z) = (z^2-1)^{-\frac{n}{2}} \int_1^z \cdots \int_1^z P_\nu(w)(dw)^n, \tag{3.26}$$

by considering the specialization $\mu = 0$ in (3.24) which follows by using (2.2) and [3, (14.9.13)]

$$P_\nu^{-n}(z) = \frac{\Gamma(\nu-n+1)}{\Gamma(\nu+n+1)} P_\nu^n(z), \quad n \in \mathbb{N}_0. \tag{3.27}$$

Remark 3.19. *Note the special value [3, (14.5.19)]*

$$P_\nu^{-\nu}(z) = \frac{(z^2-1)^{\frac{\nu}{2}}}{2^\nu \Gamma(\nu+1)}. \tag{3.28}$$

Using this special value and connection properties of associated Legendre functions we are able to derive various expression for the associated Legendre functions with the order equal to plus or minus the degree.

Corollary 3.20. *Let $\Re\nu > -\frac{1}{2}$, $z \in \mathbb{C} \setminus (-\infty, 1]$. Then*

$$P_\nu^\nu(z) = \frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} (z^2-1)^{\frac{\nu}{2}} + \frac{2^{\nu+1}}{\pi} \sin(\pi\nu)\Gamma(\nu+1)(z^2-1)^{\frac{\nu}{2}} \int_z^\infty \frac{dw}{(w^2-1)^{\nu+1}}. \tag{3.29}$$

Proof. Start with the connection relation [3, (14.9.15)]

$$P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z) + \frac{2}{\pi} \sin(\pi\mu)\Gamma(\nu+\mu+1)\boldsymbol{Q}_\nu^\mu(z),$$

then, relying on (3.28), the choice $\mu = \nu$ completes the proof. □

Remark 3.21. *Observe that if $\nu = n \in \mathbb{N}_0$ then*

$$P_n^n(z) = \frac{2^n \Gamma(n+\frac{1}{2})(z^2-1)^{\frac{n}{2}}}{\sqrt{\pi}} = (2n-1)!!(z^2-1)^{\frac{n}{2}},$$

where we have used [4, (6.1.12)], and $(\cdot)!!$ is the double factorial symbol.

Corollary 3.22. *Let $\Re\nu > 0$, $z \in \mathbb{C} \setminus (-\infty, 1]$. Then*

$$P_{-\nu}^{-\nu}(z) = \frac{1}{2^{\nu-1}\Gamma(\nu)(z^2-1)^{\frac{\nu}{2}}} \int_1^z (w^2-1)^{\nu-1} dw.$$

Proof. Starting with (3.17) and using the Whipple relation for associated Legendre functions (3.19), followed by the application of the involution [9, Section 2] $\zeta(z) := \log\coth\frac{z}{2}$, then making the change of variables $w = \zeta/\sqrt{\zeta^2-1}$ completes the proof. □

An interesting definite integral follows from the behavior of the above integral representation near the singularity at $z = 1$. Using [3, (14.9.15)], then

$$P_\nu^{-\nu}(z) = \frac{1}{\Gamma(2\nu+1)} \left(P_\nu^\nu(z) - \frac{2\sin(\pi\nu)}{\pi} \boldsymbol{Q}_\nu^\nu(z) \right). \tag{3.30}$$

After replacement of (3.17) and (3.29) in (3.28) we obtain

$$\begin{aligned}
P_\nu^{-\nu}(z) &= -\frac{\sin(\pi\nu)(z^2-1)^{\frac{\nu}{2}}}{\sqrt{\pi}\,2^{\nu-1}\Gamma(\nu+\frac{1}{2})}\left(\int_{z/\sqrt{z^2-1}}^{\infty}\frac{dw}{(w^2-1)^{-\nu+1/2}}+\int_z^{\infty}\frac{dw}{(w^2-1)^{\nu+1}}\right)\\
&= -\frac{\sin(\pi\nu)(z^2-1)^{\frac{\nu}{2}}}{\sqrt{\pi}\,2^{\nu-1}\Gamma(\nu+\frac{1}{2})}\int_1^{\infty}\frac{dw}{(w^2-1)^{\nu+1}}\\
&= \frac{(z^2-1)^{\frac{\nu}{2}}}{2^\nu\Gamma(\nu+1)}.
\end{aligned} \tag{3.31}$$

Which is simply a re-evaluation of (3.28). From the previous identities the following result follows.

Corollary 3.23. *Let* $-\frac{1}{2}\le\Re\nu\le 0$. *Then*

$$\frac{\Gamma(-\nu)\Gamma(\nu+\frac{1}{2})}{2\sqrt{\pi}}=\int_{z/\sqrt{z^2-1}}^{\infty}\frac{dw}{(w^2-1)^{-\nu+1/2}}+\int_z^{\infty}\frac{dw}{(w^2-1)^{\nu+1}}=\int_1^{\infty}\frac{dw}{(w^2-1)^{\nu+1}}.$$

Proof. The formula follows after a straightforward calculation starting from (3.31), making the change of variables $w=\zeta/\sqrt{\zeta^2-1}$ in the first integral and taking into account (3.28). ☐

4. Ferrers functions of the first and second kind

The Ferrers functions of the first and second kinds (associated Legendre functions of the first and second kinds on-the-cut) $\mathrm{P}_\nu^\mu:(-1,1)\to\mathbb{C}$ is defined in [3, (14.3.1)]

$$\mathrm{P}_\nu^\mu(x):=\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}}\frac{1}{\Gamma(1-\mu)}\,{}_2F_1\left(\begin{matrix}-\nu,\nu+1\\1-\mu\end{matrix};\frac{1-x}{2}\right), \tag{4.1}$$

$$=\frac{(1-x^2)^{\frac{\mu}{2}}}{2^\mu\Gamma(1+\mu)}\,{}_2F_1\left(\begin{matrix}\nu+\mu+1,\mu-\nu\\1+\mu\end{matrix};\frac{1-x}{2}\right), \tag{4.2}$$

where we have applied the Euler transformation (2.4), to the single summation definition of the Ferrers function of the first kind produces the second representation for the Ferrers function of the first kind. Also, $\mathrm{Q}_\nu^\mu:(-1,1)\to\mathbb{C}$ is defined in [3, (14.3.2)]

$$\begin{aligned}
\mathrm{Q}_\nu^\mu(x):=\frac{\pi}{2\sin(\pi\mu)}\Bigg[&\frac{\cos(\pi\mu)}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}}{}_2F_1\left(\begin{matrix}-\nu,\nu+1\\1-\mu\end{matrix};\frac{1-x}{2}\right)\\
&-\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}}\frac{1}{\Gamma(1+\mu)}\,{}_2F_1\left(\begin{matrix}-\nu,\nu+1\\1+\mu\end{matrix};\frac{1-x}{2}\right)\Bigg],
\end{aligned} \tag{4.3}$$

where $\mu\notin\mathbb{Z}$. However, $\mathrm{Q}_\nu^\mu(x)$ can be analytically continued for $\mu\in\mathbb{Z}$ which is demonstrated by [3, (14.3.12)],

$$\begin{aligned}
\mathrm{Q}_\nu^\mu(x)=\frac{\sqrt{\pi}\,2^{\mu-1}}{(1-x^2)^{\frac{\mu}{2}}}\Bigg[&\frac{-\sin\left(\frac{\pi}{2}(\nu+\mu)\right)\Gamma(\frac{\nu+\mu+1}{2})}{\Gamma(\frac{\nu-\mu+2}{2})}\,{}_2F_1\left(\begin{matrix}-\frac{\nu+\mu}{2},\frac{\nu-\mu+1}{2}\\\frac{1}{2}\end{matrix};x^2\right)\\
&+\frac{2\cos\left(\frac{\pi}{2}(\nu+\mu)\right)\Gamma(\frac{\nu+\mu+2}{2})\,x}{\Gamma(\frac{\nu-\mu+1}{2})}\,{}_2F_1\left(\begin{matrix}\frac{-\nu-\mu+1}{2},\frac{\nu-\mu+2}{2}\\\frac{3}{2}\end{matrix};x^2\right)\Bigg].
\end{aligned} \tag{4.4}$$

Another hypergeometric representation of the Ferrers function of the second kind which we will use below is

$$\begin{aligned}\mathsf{Q}_\nu^\mu(x) = {} & \frac{2^{\mu-1}\cos(\pi\mu)}{(1-x^2)^{\frac{\mu}{2}}}\Gamma(\mu)x^{\nu+\mu}\,{}_2F_1\!\left(\begin{matrix}\frac{-\nu-\mu}{2}, \frac{-\nu-\mu+1}{2}\\ 1-\mu\end{matrix}; \frac{x^2-1}{x^2}\right)\\ & + \frac{\Gamma(\nu+\mu+1)\Gamma(-\mu)}{2^{\mu+1}\Gamma(\nu-\mu+1)}(1-x^2)^{\frac{\mu}{2}}x^{\nu-\mu}\,{}_2F_1\!\left(\begin{matrix}\frac{\mu-\nu}{2}, \frac{\mu-\nu+1}{2}\\ \mu+1\end{matrix}; \frac{x^2-1}{x^2}\right), \qquad (4.5)\end{aligned}$$

which can be obtained by a limiting procedure [3, cf. (14.23.5)]

$$\mathsf{Q}_\nu^\mu(x) = \frac{\Gamma(\nu+\mu+1)}{2}\left(\mathrm{e}^{-\frac{1}{2}i\pi\mu}\boldsymbol{Q}_\nu^\mu(x+i0) + \mathrm{e}^{\frac{1}{2}i\pi\mu}\boldsymbol{Q}_\nu^\mu(x-i0)\right),$$

for $x \in (-1,1)$, starting from [10, Entry 29, p. 162].

4.1. *The Ferrers function of the first kind*

Here we derive interesting derivative formulae and integral representations for the Ferrers function of the first kind.

First we treat some multi-integrals of the Ferrers function of the first kind from the singularity at $x = 1$.

Remark 4.1. *Using* (4.2) *and* (2.7) *produces the following derivative formula for* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, *namely*

$$\frac{d^n}{dx^n}\frac{\mathsf{P}_\nu^{-\mu}(x)}{(1-x^2)^{\frac{\mu}{2}}} = (-1)^n(\nu+\mu+1)_n(\mu-\nu)_n\frac{\mathsf{P}_\nu^{-\mu-n}(x)}{(1-x^2)^{\frac{\mu+n}{2}}}. \qquad (4.6)$$

From (4.2) integral representations can be obtained through repeated integration. For instance, the single integral result is given as follows.

Corollary 4.2. *Let* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. *Then*

$$\int_x^1 \frac{\mathsf{P}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = \frac{1}{(\nu+\mu)(\nu-\mu+1)}\left(\frac{1}{2^{\mu-1}\Gamma(\mu)} - \frac{\mathsf{P}_\nu^{-\mu+1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}}\right). \qquad (4.7)$$

Proof. In order to derive this result, integrate (4.6) for $n = 1$ with the fundamental theorem of calculus (3.22) and taking advantage of [3, (14.8.1)]

$$\mathsf{P}_\nu^\mu(x) \sim \frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1-x}\right)^{\frac{\mu}{2}}, \qquad (4.8)$$

as $x \to 1^-$, which completes the proof. □

The above result can be generalized by repeatedly integrating the above formula.

Theorem 4.3. *Let $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. Then*

$$\int_x^1 \cdots \int_x^1 \frac{\mathsf{P}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}(dw)^n = \frac{1}{(-\nu-\mu)_n(\nu-\mu+1)_n}\Bigg(\frac{\mathsf{P}_\nu^{-\mu+n}(x)}{(1-x^2)^{\frac{\mu-n}{2}}}$$

$$-\frac{(-1)^n(-\mu)_n}{2^{\mu-n}\Gamma(\mu+1)}\sum_{k=0}^{n-1}\frac{(\nu+\mu+1-n)_k(\mu-\nu-n)_k}{k!(\mu+1-n)_k}\left(\frac{1-x}{2}\right)^k\Bigg) \tag{4.9}$$

$$= \frac{(1-x)^n}{2^\mu n!\Gamma(\mu+1)}\,{}_3F_2\left(\begin{matrix}\nu+\mu+1, \mu-\nu, 1\\ \mu+1, n+1\end{matrix}; \frac{1-x}{2}\right). \tag{4.10}$$

Proof. Repeated integration of (4.6) while noting (4.8), and using induction with with (2.1) derives the two sum expression (4.9). By rewriting the Ferrers function of the first kind on the right-hand side of (4.10) in terms of the Gauss hypergeometric representation (4.2), the finite sum term cancels the first n terms of the k sum, and rewriting the resulting expression shows it can be written in terms of a nonterminating ${}_3F_2$. This completes the proof. ☐

Remark 4.4. *It is clear that Theorem 4.3 is a generalization of [3, (14.6.6)]*

$$\mathsf{P}_\nu^{-n}(x) = (1-x^2)^{-\frac{n}{2}}\int_x^1 \cdots \int_x^1 \mathsf{P}_\nu(w)(dw)^n, \tag{4.11}$$

by considering the specialization $\mu = 0$ in (4.9), where we have used [3, (14.9.3)]

$$\mathsf{P}_\nu^{-n}(x) = (-1)^n\frac{\Gamma(\nu-n+1)}{\Gamma(\nu+n+1)}\mathsf{P}_\nu^n(x), \quad n \in \mathbb{N}_0. \tag{4.12}$$

By using the antiderivative [3, (14.17.2)]

$$\int \frac{\mathsf{P}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = \frac{1}{(\nu+\mu)(\nu-\mu+1)}\frac{\mathsf{P}_\nu^{-\mu+1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} + C, \tag{4.13}$$

where C is an arbitrary constant to derive some interesting integral representations for Ferrers functions of the first kind. Also, by using this formula to obtain a useful derivative formula for Ferrers functions of the first kind (see Remark 4.1).

Next we treat some multi-integrals of the Ferrers function of the first kind from the origin.

Evaluation of (4.13) at the endpoints of integration produces the following results.

Theorem 4.5. *Let $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. Then*

$$\int_0^x \frac{\mathsf{P}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = \frac{1}{(\nu+\mu)(\nu-\mu+1)}\left(\frac{\mathsf{P}_\nu^{-\mu+1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} - \frac{\sqrt{\pi}}{2^{\mu-1}\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)}\right). \tag{4.14}$$

Proof. Evaluating (4.13) at the endpoints of integration using [3, (14.5.1)]

$$\mathsf{P}_\nu^{-\mu}(0) = \frac{\sqrt{\pi}}{2^\mu\Gamma\left(\frac{\nu+\mu+2}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)}, \tag{4.15}$$

completes the proof. ☐

Theorem 4.6. *Let* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. *Then*

$$\int_0^x \cdots \int_0^x \frac{\mathsf{P}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}(dw)^n = (-1)^n \sum_{k=n}^{\infty} (\nu+\mu+1)_{k-n}(\mu-\nu)_{k-n}\mathsf{P}_\nu^{-\mu+n-k}(0) \tag{4.16}$$

$$\begin{aligned} &= \frac{(-1)^n}{(-\nu-\mu)_n(\nu-\mu+1)_n} \\ &\quad \times \left(\frac{\mathsf{P}_\nu^{-\mu+n}(x)}{(1-x^2)^{\frac{\mu-n}{2}}} - \frac{\sqrt{\pi}}{2^{\mu-n}} \sum_{k=0}^{n-1} \frac{(\nu+\mu-n+1)_k(\mu-\nu-n)_k\left(-\frac{x}{2}\right)^k}{k!\,\Gamma\left(\frac{\nu+\mu+2-n+k}{2}\right)\Gamma\left(\frac{\mu-\nu+1-n+k}{2}\right)} \right) \end{aligned} \tag{4.17}$$

$$\begin{aligned} &= \frac{\sqrt{\pi}\,x^n}{2^{\mu} n!\,\Gamma\left(\frac{\nu+\mu+2}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)} {}_3F_2\left({\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, 1 \atop \frac{n+1}{2}, \frac{n+2}{2}} ; x^2 \right) \\ &\quad - \frac{\sqrt{\pi}\,x^{n+1}}{2^{\mu-1}(n+1)!\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)} {}_3F_2\left({\frac{\mu-\nu+1}{2}, \frac{\nu+\mu+2}{2}, 1 \atop \frac{n+2}{2}, \frac{n+3}{2}} ; x^2 \right). \end{aligned} \tag{4.18}$$

Proof. Repeatedly applying Theorem 4.5 without evaluating $\mathsf{P}_\nu^{\mu}(0)$ and then computing the Maclaurin expansion of $\mathsf{P}_\nu^{-\mu+n}(x)/(1-x^2)^{(\mu-n)/2}$ yields the first expression. Using induction evaluating $\mathsf{P}_\nu^{-\mu+n}(0)$ with (2.1) produces the second expression. The third expression is obtained by starting with the first expression, evaluating $\mathsf{P}_\nu^{-\mu+n-k}(0)$, shifting the sum index by n and splitting the sum into even and odd parts. ☐

On the other hand, by applying the antiderivative [3, (14.17.1)]

$$\int \frac{\mathsf{P}_\nu^{\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}} dw = -\frac{\mathsf{P}_\nu^{\mu-1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} + C, \tag{4.19}$$

where C is an arbitrary constant to derive some interesting integral representations for Ferrers functions of the first kind. Also, utilizing this formula to obtain a useful derivative formula for Ferrers functions of the first kind.

Remark 4.7. *Differentiating the above result produces the following formula for* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$,

$$\frac{d^n}{dx^n} \frac{\mathsf{P}_\nu^{\mu}(x)}{(1-x^2)^{\frac{\mu}{2}}} = \frac{(-1)^n \mathsf{P}_\nu^{\mu+n}(x)}{(1-x^2)^{\frac{\mu+n}{2}}}. \tag{4.20}$$

Evaluation of (4.19) at the endpoints of integration produces the following results.

Theorem 4.8. *Let* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, $\Re\mu > 0$. *Then*

$$\int_x^1 (1-w^2)^{\frac{\mu}{2}} \mathsf{P}_\nu^{-\mu}(w) dw = (1-x^2)^{\frac{\mu+1}{2}} \mathsf{P}_\nu^{-\mu-1}(x). \tag{4.21}$$

Proof. In order to derive this result integrate (4.20) for $n = 1$ with the fundamental theorem of calculus (3.22) and taking advantage of cf. (4.8)

$$(1-x^2)^{\frac{\mu}{2}}\mathsf{P}_\nu^{-\mu}(x) \sim 0,$$

as $x \to 1^-$. This completes the proof. □

Theorem 4.9. *Let* $x \in (-1,1)$, $\nu,\mu \in \mathbb{C}$, $\Re\mu > 0$. *Then*

$$\int_x^1 \cdots \int_x^1 (1-w^2)^{\frac{\mu}{2}}\mathsf{P}_\nu^{-\mu}(w)(dw)^n = (1-x^2)^{\frac{\mu+n}{2}}\mathsf{P}_\nu^{-\mu-n}(x). \tag{4.22}$$

Proof. Repeatedly applying Theorem 4.8 through induction proves the result. □

Theorem 4.10. *Let* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu,\mu \in \mathbb{C}$. *Then*

$$\int_0^x \frac{\mathsf{P}_\nu^\mu(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = -\frac{\mathsf{P}_\nu^{\mu-1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} + \frac{2^{\mu-1}\sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu+3}{2}\right)\Gamma\left(\frac{-\nu-\mu+2}{2}\right)}. \tag{4.23}$$

Proof. Evaluating (4.13) at the endpoints of integration using (4.15) completes the proof. □

Theorem 4.11. *Let* $x \in (-1,1)$, $\nu,\mu \in \mathbb{C}$. *Then*

$$\begin{aligned}
\int_0^x \cdots \int_0^x \frac{\mathsf{P}_\nu^\mu(w)}{(1-w^2)^{\frac{\mu}{2}}}(dw)^n &= (-1)^n \sum_{k=n}^{\infty} \frac{(-x)^k}{k!}\mathsf{P}_\nu^{\mu-n+k}(0) \\
&=(-1)^n\left(\frac{\mathsf{P}_\nu^{\mu-n}(x)}{(1-x^2)^{\frac{\mu-n}{2}}} - 2^{\mu-n}\sqrt{\pi}\sum_{k=0}^{n-1}\frac{(-2x)^k}{k!\,\Gamma\left(\frac{\nu-\mu+2+n-k}{2}\right)\Gamma\left(\frac{-\nu-\mu+1+n-k}{2}\right)}\right) \\
&=\frac{\sqrt{\pi}\,2^\mu x^n}{n!\Gamma\left(\frac{\nu-\mu+2}{2}\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\mu-\nu}{2},\frac{\nu+\mu+1}{2},1\\ \frac{n+1}{2},\frac{n+2}{2}\end{matrix};x^2\right) \\
&\quad-\frac{\sqrt{\pi}\,2^{\mu+1}x^{n+1}}{(n+1)!\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{-\nu-\mu}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\mu-\nu+1}{2},\frac{\nu+\mu+2}{2},1\\ \frac{n+2}{2},\frac{n+3}{2}\end{matrix};x^2\right).
\end{aligned} \tag{4.24}$$

Proof. Repeatedly applying Theorem 4.10 without evaluating $\mathsf{P}_\nu^\mu(0)$ and then computing the Maclaurin expansion of $\mathsf{P}_\nu^{\mu-n}(x)/(1-x^2)^{(\mu-n)/2}$ yields the first expression. Using induction evaluating $\mathsf{P}_\nu^{\mu-n}(0)$ with (2.1) produces the second expression. The third expression is obtained by starting with the first expression, evaluating $\mathsf{P}_\nu^{\mu-n+k}(0)$, shifting the sum index by n and splitting the sum into even and odd parts. □

A definite-integral result near the singularity at $x = 1$ follows using (4.53), (4.54), and (2.5), namely

$$\int_0^1 (1-w^2)^{\nu-1}dw = \frac{\sqrt{\pi}\,\Gamma(\nu)}{2\Gamma(\nu+\frac{1}{2})},$$

for $\Re\nu > 0$. The well-known special value (see [3, (14.5.18)])

$$\mathsf{P}_\nu^{-\nu}(x) = \frac{(1-x^2)^{\frac{\nu}{2}}}{2^\nu \Gamma(\nu+1)}, \tag{4.25}$$

in conjunction with [11, (8.737.1)], yields the following integral representation.

Corollary 4.12. *Let $\nu \in \mathbb{C}$, $x \in (-1,1)$. Then*

$$\mathsf{P}_\nu^{\nu}(x) = \frac{2^\nu (1-x^2)^{\frac{\nu}{2}}}{\sqrt{\pi}} \left(\Gamma(\nu+\tfrac{1}{2})\cos(\pi\nu) + \frac{2\Gamma(\nu+1)}{\sqrt{\pi}} \sin(\pi\nu) \int_0^x \frac{dw}{(1-w^2)^{\nu+1}} \right). \tag{4.26}$$

Proof. Start with [3, (14.9.2)]

$$\mathsf{P}_\nu^{\mu}(x) = \cos(\pi\mu) \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \mathsf{P}_\nu^{-\mu}(x) + \frac{2}{\pi} \sin(\pi\mu) \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \mathsf{Q}_\nu^{-\mu}(x),$$

replace $\mu = \nu$, then using (4.54), (4.25) completes the proof. ☐

Remark 4.13. *Note that if $\nu = n \in \mathbb{N}_0$ then*

$$\mathsf{P}_n^{n}(x) = \frac{(-2)^n \Gamma(n+\frac{1}{2})(1-x^2)^{\frac{n}{2}}}{\sqrt{\pi}} = (-1)^n (2n-1)!!(1-x^2)^{\frac{n}{2}}, \tag{4.27}$$

where we have used [4, (6.1.12)].

4.2. The Ferrers function of the second kind

The Ferrers function of the second kind (associated Legendre function of the second kind on-the-cut) $\mathsf{Q}_\nu^\mu : (-1,1) \to \mathbb{C}$ is defined in (4.3).

First we treat some multi-integrals of the Ferrers function of the second kind to the singularity at $x = 1$.

Lemma 4.14. *Let $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$ such that $\mu \notin -\mathbb{N}$, $\nu - \mu \notin -\mathbb{N}_0$. Then*

$$\int_x^1 (1-w^2)^{\frac{\mu}{2}} \mathsf{Q}_\nu^{-\mu}(w)dw = (1-x^2)^{\frac{\mu+1}{2}} \mathsf{Q}_\nu^{-\mu-1}(x) - \frac{2^\mu \Gamma(\mu+1)\Gamma(\nu-\mu)}{\Gamma(\nu+\mu+2)}. \tag{4.28}$$

Proof. The Ferrers function of the second kind as x approaches the singularity at $x = 1$ has the following behavior [3, (14.8.6)]

$$(1-x^2)^{\frac{\mu}{2}} \mathsf{Q}_\nu^{-\mu}(x) \sim \frac{2^{\mu-1}\Gamma(\mu)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}, \tag{4.29}$$

as $x \to 1^-$, $\Re\mu > 0$. Evaluating [3, (14.17.1)] using the Ferrers function of the second kind at the endpoints of integration noting the above behavior at $x \approx 1$ completes the proof. ☐

Remark 4.15. *Applying the fundamental theorem of calculus (3.22) to Lemma 4.14 produces the following derivative formula for $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, namely*

$$\frac{d^n}{dx^n} (1-x^2)^{\frac{\mu}{2}} \mathsf{Q}_\nu^{-\mu}(x) = (-1)^n (1-x^2)^{\frac{\mu-n}{2}} \mathsf{Q}_\nu^{-\mu+n}(x). \tag{4.30}$$

Theorem 4.16. *Let* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. *Then*

$$\begin{aligned}\int_x^1 \cdots \int_x^1 (1-w^2)^{\frac{\mu}{2}} \mathsf{Q}_\nu^{-\mu}(w)(dw)^n &= (1-x^2)^{\frac{\mu+n}{2}} \mathsf{Q}_\nu^{-\mu-n}(x) \\ &\quad - \frac{(-1)^n 2^{\mu+n-1}\Gamma(\mu)\Gamma(\nu-\mu+1)(\mu)_n}{\Gamma(\nu+\mu+1)(\mu-\nu)_n(\nu+\mu+1)_n} \sum_{k=0}^{n-1} \frac{(\nu-\mu-n+1)_k(-\nu-\mu-n)_k}{k!(-\mu-n+1)_k} \left(\frac{1-x}{2}\right)^k \qquad (4.31)\\ &= -\frac{\pi}{2}\cot(\pi\mu)(1-x^2)^{\frac{\mu+n}{2}} \mathsf{P}_\nu^{-\mu-n}(x) \\ &\qquad + \frac{2^{\mu-1}\Gamma(\mu)\Gamma(\nu-\mu+1)(1-x)^n}{n!\,\Gamma(\nu+\mu+1)}\, {}_3F_2\left(\begin{matrix}\nu-\mu+1, -\nu-\mu, 1 \\ n+1, 1-\mu\end{matrix}; \frac{1-x}{2}\right). \qquad (4.32)\end{aligned}$$

Proof. Repeatedly applying Lemma 4.14 to itself using induction with (2.1) produces the first formula. The second formula is obtained by rewriting the finite sum as a sum from 0 to ∞ and subtracting the sum from n to ∞, and then finally utilizing (4.3). ☐

Remark 4.17. *Taking the* $\mu \to 0$ *limit in Corollary 4.16 produces the following multi-integration result for* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, *namely*

$$\begin{aligned}\int_x^1 \cdots \int_x^1 \mathsf{Q}_\nu(w)(dw)^n &= (1-x^2)^{\frac{n}{2}} \mathsf{Q}_\nu^{-n}(x) \\ &\quad - \frac{(-1)^n 2^{n-1}(n-1)!}{(-\nu)_n(\nu+1)_n} \sum_{k=0}^{n-1} \frac{(\nu-n+1)_k(-\nu-n)_k}{k!(1-n)_k}\left(\frac{1-x}{2}\right)^k. \qquad (4.33)\end{aligned}$$

Now we present a similar result for the Ferrers function of the second kind with order μ instead of $-\mu$.

Lemma 4.18. *Let* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$ *such that* $\mu \notin -\mathbb{N}$, $\nu - \mu \notin -\mathbb{N}_0$. *Then*

$$\int_x^1 (1-w^2)^{\frac{\mu}{2}} \mathsf{Q}_\nu^{\mu}(w)dw = \frac{1}{(\mu-\nu)(\nu+\mu+1)}\left((1-x^2)^{\frac{\mu+1}{2}}\mathsf{Q}_\nu^{\mu+1}(x) + \frac{2^\mu \pi \Gamma(\mu+1)}{\Gamma(-\mu-\frac{3}{2})\Gamma(\mu+\frac{5}{2})}\right). \qquad (4.34)$$

Proof. The Ferrers function of the second kind as x approaches the singularity at $x=1$ has the following behavior (cf. [3, (14.8.4)])

$$(1-x^2)^{\frac{\mu}{2}}\mathsf{Q}_\nu^{\mu}(x) \sim -\frac{2^{\mu-1}\pi\Gamma(\mu)}{\Gamma(-\mu-\frac{1}{2})\Gamma(\mu+\frac{3}{2})}, \qquad (4.35)$$

as $x \to 1^-$, $\Re\mu > 0$. Evaluating [3, (14.17.2)] using the Ferrers function of the second kind at the endpoints of integration noting the above behavior at $x \approx 1$ completes the proof. ☐

Remark 4.19. *Applying the fundamental theorem of calculus (3.22) to Lemma 4.18 produces the following derivative formula for* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, *namely*

$$\frac{d^n}{dx^n}(1-x^2)^{\frac{\mu}{2}}\mathsf{Q}_\nu^{\mu}(x) = (-1)^n(\nu-\mu+1)_n(-\nu-\mu)_n(1-x^2)^{\frac{\mu-n}{2}}\mathsf{Q}_\nu^{\mu-n}(x). \qquad (4.36)$$

Theorem 4.20. *Let $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. Then*

$$\begin{aligned}\int_x^1 \cdots \int_x^1 (1-w^2)^{\frac{\mu}{2}} \mathrm{Q}_\nu^\mu(w)(dw)^n &= \frac{1}{(\mu-\nu)_n(\nu+\mu+1)_n}\Bigg((1-x^2)^{\frac{\mu+n}{2}} \mathrm{Q}_\nu^{\mu+n}(x) \\ &+(-1)^{n-1}2^{\mu+n-1}\cos(\pi\mu)\Gamma(\mu+n)\sum_{k=0}^{n-1}\frac{(\nu-\mu-n+1)_k(-\nu-\mu-n)_k}{k!(-\mu-n+1)_k}\left(\frac{1-x}{2}\right)^k\Bigg) \qquad (4.37)\\ &= -\frac{\pi\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)\sin(\pi\mu)}(1-x^2)^{\frac{\mu+n}{2}}\mathrm{P}_\nu^{-\mu-n}(x) \\ &+\frac{2^{\mu-1}\cos(\pi\mu)\Gamma(\mu)}{n!}(1-x)^n {}_3F_2\left(\begin{matrix}\nu-\mu+1, -\nu-\mu, 1\\ n+1, 1-\mu\end{matrix}; \frac{1-x}{2}\right). \qquad (4.38)\end{aligned}$$

Proof. Repeatedly applying Lemma 4.18 to itself using induction with (2.1) produces the first formula. The second formula is obtained by rewriting the finite sum as a sum from 0 to ∞ and subtracting the sum from n to ∞, and then finally utilizing (4.3). ☐

Remark 4.21. *An interesting discussion is concerning whether Lemma 2.3 might be used to obtain new generalized hypergeometric representations for Corollaries 4.16, 4.20. In order to do this, one must compute the one-sided Taylor expansions of the relevant functions about the singular point $x = 1$ (the relevant functions are well-behaved at this singular point). This is readily possible, but is not practical due to the fact that the behavior of the functions in question near the singularity changes in form depending on whether $\Re\mu \lesseqgtr 0$ (see (4.29), (4.35)). The derivative terms in the Taylor series necessarily cross the $\Re\mu = 0$ boundary, so a simple result from this Lemma does not seem to be practical.*

Remark 4.22. *Taking $\mu \to 0$ limit in Corollary 4.20 produces the following multi-integration result for $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, namely*

$$\begin{aligned}\int_x^1 \cdots \int_x^1 \mathrm{Q}_\nu(w)(dw)^n &= \frac{1}{(-\nu)_n(\nu+1)_n}\Bigg((1-x^2)^{\frac{n}{2}}\mathrm{Q}_\nu^n(x) \\ &+(-1)^{n-1}2^{n-1}(n-1)!\sum_{k=0}^{n-1}\frac{(\nu-n+1)_k(-\nu-n)_k}{k!(1-n)_k}\left(\frac{1-x}{2}\right)^k\Bigg). \qquad (4.39)\end{aligned}$$

Remark 4.23. *Note that in Corollaries 4.16, 4.20, it is tempting to consider the $\mu \to 0$ limit using their ${}_3F_2$ representations. However, to zeroth order in μ, the limits cancel. One must then determine a first order approximation in μ to determine the limit behavior. After performing this calculation in both of these situations, then it turns out that the result is given in terms of the sum of several double hypergeometric series of Kampé de Fériet type (see e.g., [12, p. 27]). Since this result is very cumbersome and doesn't really shed much light on these limits, we have instead presented the above Remarks 4.17, 4.22. It should also be pointed out regarding the fact that there does not seem to be an analogous formula for the Ferrers function of the second kind in the classical list [3, (14.6.6-8)], the above reasoning most likely explains this fact. The conclusion is that any formula for negative integer order Ferrers functions of the second kind will involve a finite sum of polynomial terms in addition to the multi-integral of $\mathrm{Q}_\nu(x)$, as indicated in Remarks 4.17, 4.22.*

Next we treat some multi-integrals of the Ferrers function of the second kind from the origin.

Remark 4.24. *Applying the fundamental theorem of calculus* (3.22) *to* [3, (14.17.2)] *produces the following derivative formula for* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, *namely*

$$\frac{d^n}{dx^n}\frac{\mathsf{Q}_\nu^{-\mu}(x)}{(1-x^2)^{\frac{\mu}{2}}} = (-1)^n(\nu+\mu+1)_n(\mu-\nu)_n\frac{\mathsf{Q}_\nu^{-\mu-n}(x)}{(1-x^2)^{\frac{\mu+n}{2}}}. \tag{4.40}$$

Theorem 4.25. *Let* $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. *Then*

$$\int_0^x \frac{\mathsf{Q}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = \frac{1}{(\nu+\mu)(\nu-\mu+1)}\left(\frac{\mathsf{Q}_\nu^{-\mu+1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} - \frac{\pi^{\frac{3}{2}}\Gamma\left(\frac{\nu-\mu+2}{2}\right)}{2^\mu\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\mu-\nu-1}{2}\right)\Gamma\left(\frac{\nu-\mu+3}{2}\right)}\right). \tag{4.41}$$

Proof. Evaluating [3, (14.17.2)] (expressed as a Ferrers function of the second kind) at the endpoints of integration using [3, (14.5.3)]

$$\mathsf{Q}_\nu^{-\mu}(0) = \frac{\pi^{\frac{3}{2}}\Gamma\left(\frac{\nu-\mu+1}{2}\right)}{2^{\mu+1}\Gamma\left(\frac{\nu+\mu+2}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}, \tag{4.42}$$

completes the proof. ☐

Theorem 4.26. *Let* $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. *Then*

$$\int_0^x \cdots \int_0^x \frac{\mathsf{Q}_\nu^{-\mu}(w)}{(1-w^2)^{\frac{\mu}{2}}}(dw)^n = (-1)^n\sum_{k=n}^\infty \frac{(-x)^k}{k!}(\nu+\mu+1)_{k-n}(\mu-\nu)_{k-n}\mathsf{Q}_\nu^{-\mu+n-k}(0) \tag{4.43}$$

$$= \frac{(-1)^n}{(-\nu-\mu)_n(\nu-\mu+1)_n}\left(\frac{\mathsf{Q}_\nu^{-\mu+n}(x)}{(1-x^2)^{\frac{\mu-n}{2}}}\right.$$

$$\left. - \frac{\pi^{\frac{3}{2}}}{2^{\mu-n+1}}\sum_{k=0}^{n-1}\frac{(\nu+\mu+1-n)_k(\mu-\nu-n)_k\Gamma\left(\frac{\nu-\mu+1+n-k}{2}\right)\left(-\frac{x}{2}\right)^k}{k!\,\Gamma\left(\frac{\nu+\mu+2-n+k}{2}\right)\Gamma\left(\frac{\mu-\nu-n+k}{2}\right)\Gamma\left(\frac{\nu-\mu+2+n-k}{2}\right)}\right) \tag{4.44}$$

$$= \frac{\pi^{\frac{3}{2}}x^n\Gamma\left(\frac{\nu-\mu+1}{2}\right)}{n!2^{\mu+1}\Gamma\left(\frac{\nu+\mu+2}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\nu+\mu+1}{2}, \frac{\mu-\nu}{2}, 1\\ \frac{n+1}{2}, \frac{n+2}{2}\end{matrix}; x^2\right)$$

$$+\frac{\pi^{\frac{3}{2}}x^{n+1}\Gamma\left(\frac{\nu-\mu+2}{2}\right)}{(n+1)!2^\mu\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+1}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\nu+\mu+2}{2}, \frac{\mu-\nu+1}{2}, 1\\ \frac{n+2}{2}, \frac{n+3}{2}\end{matrix}; x^2\right). \tag{4.45}$$

Proof. Repeatedly applying Theorem 4.25 without evaluating $\mathsf{Q}_\nu^{-\mu}(0)$ and then computing the Maclaurin expansion of $\mathsf{Q}_\nu^{-\mu+n}(x)/(1-x^2)^{(\mu-n)/2}$ yields the first expression. Using induction evaluating $\mathsf{Q}_\nu^{-\mu+n}(0)$ with (2.1) produces the second expression. The third expression is obtained by starting with the first expression, evaluating $\mathsf{Q}_\nu^{-\mu+n-k}(0)$, shifting the sum index by n and splitting the sum into even and odd parts. ☐

Remark 4.27. *Taking the limit as $\mu = 0$ in Theorem 4.26 produces the following multi-integration result, namely for $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, then*

$$\int_0^x \cdots \int_0^x \mathrm{Q}_\nu(w)(dw)^n = \frac{1}{(-\nu)_n(\nu+1)_n}\Bigg((-1)^n(1-x^2)^{\frac{n}{2}}\mathrm{Q}_\nu^n(x)$$
$$+(-1)^{n+1}2^{n-1}\pi^{\frac{3}{2}}\sum_{k=0}^{n-1}\frac{(\nu+1-n)_k(-\nu-n)_k\Gamma\left(\frac{\nu+1+n-k}{2}\right)\left(-\frac{x}{2}\right)^k}{k!\,\Gamma\left(\frac{\nu+2-n+k}{2}\right)\Gamma\left(\frac{-\nu-n+k}{2}\right)\Gamma\left(\frac{\nu+2+n-k}{2}\right)}\Bigg). \tag{4.46}$$

On the other hand, by applying the antiderivative [3, (14.17.1)]

$$\int \frac{\mathrm{Q}_\nu^\mu(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = -\frac{\mathrm{Q}_\nu^{\mu-1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} + C, \tag{4.47}$$

where C is an arbitrary constant to derive some interesting integral representations for Ferrers functions of the second kind.

An examination of the above formula (4.47) produces the following results. For instance, by applying this formula to obtain a useful derivative formula for Ferrers functions of the second kind.

Remark 4.28. *Differentiating the above formula with (2.1) produces the following formula for $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$,*

$$\frac{d^n}{dx^n}\frac{\mathrm{Q}_\nu^\mu(x)}{(1-x^2)^{\frac{\mu}{2}}} = \frac{(-1)^n\mathrm{Q}_\nu^{\mu+n}(x)}{(1-x^2)^{\frac{\mu+n}{2}}}. \tag{4.48}$$

Theorem 4.29. *Let $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$, such that $\nu+\mu \notin -2\mathbb{N}_0$. Then*

$$\int_0^x \frac{\mathrm{Q}_\nu^\mu(w)}{(1-w^2)^{\frac{\mu}{2}}}dw = -\frac{\mathrm{Q}_\nu^{\mu-1}(x)}{(1-x^2)^{\frac{\mu-1}{2}}} - \frac{2^{\mu-2}\pi^{\frac{3}{2}}\Gamma\left(\frac{\nu+\mu}{2}\right)}{\Gamma\left(\frac{\nu-\mu+3}{2}\right)\Gamma\left(\frac{-\nu-\mu-1}{2}\right)\Gamma\left(\frac{\nu+\mu+3}{2}\right)}. \tag{4.49}$$

Proof. Evaluating (4.47) at the endpoints of integration using (4.42) completes the proof. □

Theorem 4.30. *Let $n \in \mathbb{N}_0$, $x \in (-1,1)$, $\nu, \mu \in \mathbb{C}$. Then*

$$\int_0^x \cdots \int_0^x \frac{\mathrm{Q}_\nu^\mu(w)}{(1-w^2)^{\frac{\mu}{2}}}(dw)^n = (-1)^n\sum_{k=n}^{\infty}\frac{(-x)^k}{k!}\mathrm{Q}_\nu^{\mu-n+k}(0) \tag{4.50}$$

$$= \frac{(-1)^n\mathrm{Q}_\nu^{\mu-n}(x)}{(1-x^2)^{\frac{\mu-n}{2}}} + \frac{(-1)^n\pi^{\frac{3}{2}}}{2^{n+1-\mu}}\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{\nu+\mu+1-n+k}{2}\right)(-2x)^k}{k!\,\Gamma\left(\frac{\nu-\mu+2+n-k}{2}\right)\Gamma\left(\frac{-\nu-\mu-2+n-k}{2}\right)\Gamma\left(\frac{\nu+\mu+4-n+k}{2}\right)}$$
$$= \frac{\pi^{\frac{3}{2}}\,2^{\mu-1}x^n\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{n!\Gamma\left(\frac{\nu-\mu+2}{2}\right)\Gamma\left(\frac{-\nu-\mu}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, 1\\ \frac{n+1}{2}, \frac{n+2}{2}\end{matrix}; x^2\right)$$
$$-\frac{\pi^{\frac{3}{2}}\,2^{\mu}x^{n+1}\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{(n+1)!\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{-\nu-\mu-1}{2}\right)\Gamma\left(\frac{\nu+\mu+3}{2}\right)}\,{}_3F_2\left(\begin{matrix}\frac{\mu-\nu+1}{2}, \frac{\nu+\mu+2}{2}, 1\\ \frac{n+2}{2}, \frac{n+3}{2}\end{matrix}; x^2\right). \tag{4.51}$$

Proof. Repeatedly applying Theorem 4.29 without evaluating $\mathsf{Q}_\nu^\mu(0)$ and then computing the Maclaurin expansion of $\mathsf{Q}_\nu^{\mu-n}(x)/(1-x^2)^{(\mu-n)/2}$ yields the first expression. Using induction evaluating $\mathsf{Q}_\nu^{\mu-n}(0)$ with (2.1) produces the second expression. The third expression is obtained by starting with the first expression, evaluating $\mathsf{Q}_\nu^{\mu-n+k}(0)$, shifting the sum index by n and splitting the sum into even and odd parts. ☐

Theorem 4.31. *Let $\nu \in \mathbb{C}$. Then,*

$$\int \frac{dx}{(1-x^2)^{\nu+1}} = x\,{}_2F_1\left(\begin{matrix}\frac12, \nu+1\\ \frac32\end{matrix};x^2\right) + C = \frac{2^\nu\Gamma(\nu+\frac12)}{\sqrt{\pi}\,(1-x^2)^{\frac{\nu}{2}}}\mathsf{Q}_\nu^{-\nu}(x) + C, \tag{4.52}$$

where C is an arbitrary constant.

Proof. The Gauss hypergeometric function in the antiderivative follows using (2.7), (2.8), (2.9), as in the proof of Theorem 3.12, with the Ferrers function following directly using

$$\mathsf{Q}_\nu^{-\nu}(x) = \frac{\sqrt{\pi}\,x(1-x^2)^{\frac{\nu}{2}}}{2^\nu\Gamma(\nu+\frac12)}\,{}_2F_1\left(\begin{matrix}\frac12, \nu+1\\ \frac32\end{matrix};x^2\right), \tag{4.53}$$

which follows from (4.4). This completes the proof. ☐

The following very simple integral representation for the Ferrers function of the second kind is a consequence of Theorem 4.31.

Corollary 4.32. *Let $\nu \in \mathbb{C}$, $x \in (-1,1)$. Then*

$$\mathsf{Q}_\nu^{-\nu}(x) = \frac{\sqrt{\pi}(1-x^2)^{\frac{\nu}{2}}}{2^\nu\,\Gamma(\nu+\frac12)}\int_0^x \frac{dw}{(1-w^2)^{\nu+1}}. \tag{4.54}$$

Proof. Evaluating the antiderivative Theorem 4.31 at the endpoints of integration completes the proof. ☐

Remark 4.33. *One can also show that Corollary 4.32 also follows directly from Theorem 4.25. This is true even though Theorem 4.25 is not strictly valid for $\mu = -\nu$. The result can be obtained by taking the limit as $\mu \to -\nu$ in Theorem 4.25 and using the Gauss hypergeometric representation of the Ferrers function of the second kind with argument $(1-x^{-2})$, namely (4.5).*

Corollary 4.34. *Let $\nu \in \mathbb{C}$, $x \in (-1,1)$. Then*

$$\begin{aligned}\mathsf{Q}_\nu^{\nu}(x) &= -2^{\nu-1}\sqrt{\pi}\,\Gamma(\nu+\tfrac12)\sin(\pi\nu)(1-x^2)^{\frac{\nu}{2}}\\ &\qquad +2^\nu\Gamma(\nu+1)\cos(\pi\nu)(1-x^2)^{\frac{\nu}{2}}\int_0^x \frac{dw}{(1-w^2)^{\nu+1}}.\end{aligned} \tag{4.55}$$

Proof. Using the connection relation [10, p. 170]

$$\mathsf{Q}_\nu^{-\mu}(x) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(\cos(\pi\mu)\mathsf{Q}_\nu^\mu(x) + \frac{\pi}{2}\sin(\pi\mu)\mathsf{P}_\nu^\mu(x)\right),$$

setting $\mu = \nu$, and using the above results completes the proof. ☐

Remark 4.35. *Note that if* $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}_0$ *then Corollary 4.34 reduces to the following special value*

$$\mathrm{Q}_{n+\frac{1}{2}}^{n+\frac{1}{2}}(x) = (-1)^{n+1} 2^{n-\frac{1}{2}} n! \sqrt{\pi}\, (1-x^2)^{\frac{n}{2}+\frac{1}{4}}. \tag{4.56}$$

5. Conclusion

In this paper, we explore some implications of the existence of multi-derivative formulae for associated Legendre functions of the first and second kinds P_ν^μ, $\boldsymbol{Q}_\nu^\mu$, and Ferrers functions of the first and second kinds P_ν^μ, Q_ν^μ. These multi-derivative formulae (see Remarks 3.9, 3.15, 4.1, 4.7, 4.15, 4.19, 4.24, 4.28) have the useful property that the degree (ν) is left unchanged by the multi-derivative. The order (μ) is then shifted by unit increments depending on the number of derivatives. These multi-derivative formulae generalize some classical multi-derivative formulae for these functions with integer order [3, (14.6.1)–(14.6.5)].

Due to the existence of these multi-derivative formulae, and certain special known values and limiting behaviors near the singularities of these functions, we derive several multi-integral representations for these functions. These multi-integral representations are shown to be given either in terms of (i) a sum of two ${}_3F_2$'s (Theorems 4.6, 4.11, 4.26, 4.30); (ii) a ${}_2F_1$ and a ${}_3F_2$ (Theorems 4.16, 4.20); (iii) a single ${}_3F_2$ (Theorems 3.17, 4.3); or (iv) a single ${}_2F_1$ (Theorems 3.10, 4.11). These multi-integral representations generalize some classical multi-integrals for these functions with integer order [3, (14.6.6)–(14.6.8)].

As mentioned in the introduction, many of the functions encountered in this work represent fundamental solutions for the Laplace-Beltrami operator on Riemannian manifolds of constant curvature. Multi-integrals and derivatives of these functions are essential in performing global analysis for these fundamental solutions on these manifolds. One interesting open problem where this work is almost certainly essential is for obtaining fundamental solutions of natural powers of the Laplace-Beltrami operator (polyharmonic) on these manifolds. This analysis will be investigated in future publications.

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References

1. Cohl, H.S. Opposite Antipodal Fundamental Solution of Laplace's Equation in Hyperspherical Geometry. *Symmetry, Integrability and Geometry: Methods and Applications* **2011**, *7:108*, 14.
2. Cohl, H.S.; Kalnins, E.G. Fourier and Gegenbauer expansions for a fundamental solution of the Laplacian in the hyperboloid model of hyperbolic geometry. *Journal of Physics A: Mathematical and Theoretical* **2012**, **45**:*14*, 145206.
3. *NIST Digital Library of Mathematical Functions*. *http://dlmf.nist.gov*, Release 1.0.27 of 2020-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
4. Abramowitz, M.; Stegun, I.A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*; Vol. 55, *National Bureau of Standards Applied Mathematics Series*, U.S. Government Printing Office, Washington, D.C., 1972; pp. xiv+1046.
5. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions. Vol. I*; Robert E. Krieger Publishing Co. Inc.: Melbourne, Fla., 1981; pp. xiii+302.
6. Cohl, H.S.; Dang, T.H.; Dunster, T.M. Fundamental Solutions and Gegenbauer Expansions of Helmholtz Operators in Riemannian Spaces of Constant Curvature. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), Special Issue on Orthogonal Polynomials, Special Functions and Applications (OPSFA14)* **2018**, *14:136*, 45.

7. Andrews, G.E.; Askey, R.; Roy, R. *Special functions*; Vol. 71, *Encyclopedia of Mathematics and its Applications*, Cambridge University Press: Cambridge, 1999; pp. xvi+664.
8. Olver, F.W.J. *Asymptotics and Special Functions*; AKP Classics, A K Peters Ltd.: Wellesley, MA, 1997; pp. xviii+572. Reprint of the 1974 original [Academic Press, New York].
9. Cohl, H.S. On parameter differentiation for integral representations of associated Legendre functions. *Symmetry, Integrability and Geometry: Methods and Applications, Special Issue on Symmetry, Separation, Super-integrability and Special Functions (S^4)* **2011**, **7**:*050*, 16.
10. Magnus, W.; Oberhettinger, F.; Soni, R.P. *Formulas and theorems for the special functions of mathematical physics*; Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966; pp. viii+508.
11. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, seventh ed.; Elsevier/Academic Press, Amsterdam, 2007; pp. xlviii+1171.
12. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian hypergeometric series*; Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd.: Chichester, 1985; p. 425.