SECOND STRUCTURE RELATION FOR q-SEMICLASSICAL POLYNOMIALS OF THE HAHN TABLEAU

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ABSTRACT. q-Classical orthogonal polynomials of the q-Hahn tableau are characterized from their orthogonality condition and by a first and a second structure relation. Unfortunately, for the q-semiclassical orthogonal polynomials (a generalization of the classical ones) we find only in the literature the first structure relation. In this paper, a second structure relation is deduced. In particular, by means of a general finite-type relation between a q-semiclassical polynomial sequence and the sequence of its q-differences such a structure relation is obtained.

1. INTRODUCTION

The q-Classical orthogonal polynomial sequences (Big q-Jacobi, q-Laguerre, Al-Salam Carlitz I, q-Charlier, etc.) are characterized by the property that the sequence of its monic q-difference polynomials is, again, orthogonal (Hahn's property, see [?]). In fact, the q-difference operator is a particular case of the Hahn operator which is defined as follows

$$L_{q,\omega}(f)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, \qquad \omega \in \mathbb{C},\ q \in \mathbb{C},\ |q| \neq 1.$$

In the sequel, we are going to work with q-semiclassical orthogonal polynomials and q-classical polynomials of the Hahn Tableau, hence we will consider the q-linear lattice $x(s)$, i.e. $x(s+1) = qx(s) + \omega$. Therefore, for the sake of convenience we will denote $\Delta^{(1)} \equiv L_{q,\omega}$. Notice that for $q = 1$ we get the forward difference operator Δ. In such a case, when $w \to 0$ we recover the standard semiclassical orthogonal polynomials [?].

Taking into account the role of such families of q-polynomials in the analysis of hypergeometric q-difference equations resulting from physical problems as the q-Schrödinger equation, q-harmonic oscillators, the connection and the linearization problems among others there is an increasing interest to study them. Moreover, the connection between the representation theory of quantum algebras and the q-orthogonal polynomials is well known (see [?] and references therein).

We also find many different approaches to the subject in the literature. For instance, the functional equation (the so-called Pearson equation) satisfied by the corresponding moment functionals allows an efficient study of some properties of q-classical polynomials [?], [?], [?], [?]. However, the q-classical sequences of orthogonal polynomials $\{\mathrm{C}_n\}_{n\geq 0}$ can also be characterized taking into account its orthogonality as well as one of the two following difference equations, the so-called structure relations.

- *First structure relation* [?], [?], [?]

$$\Phi(s)C_n^{[1]}(s) = \sum_{\nu=n}^{n+t} \lambda_{n,\nu}\mathrm{C}_\nu(s),\ n \geq 0, \qquad \lambda_{n,n} \neq 0,\ n \geq 0, \tag{1}$$

where Φ is a polynomial with $\deg \Phi = t \leq 2$ and $C_n^{[1]}(s) := [n+1]^{-1}\Delta^{(1)}\mathrm{C}_{n+1}(s)$, being

$$[n] := (q^n - 1)/(q-1), \quad n \geq 0.$$

- •• *Second structure relation* [?, ?]

$$\mathrm{C}_n(s) = \sum_{\nu=n-t}^{n} \theta_{n,\nu}C_\nu^{[1]}(s), \qquad n \geq t,\ 0 \leq t \leq 2, \quad \theta_{n,n} = 1,\ n \geq t. \tag{2}$$

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The q-classical orthogonal polynomials were introduced by W. Hahn [?] and also analyzed in [?]. The generalization of this families leads to q-semiclassical orthogonal polynomials which were introduced by P. Maroni and extensively studied in the last decade by himself, L. Kheriji, J. C. Medem, and others (see [?, ?]).
For q-classical orthogonal polynomial sequences, which are q-semiclassical of class zero, the structure relations (1) and (2) become

$$\phi(s)L_{q,\omega}P_n(s) = \widetilde{\alpha}_n P_{n+1}(s) + \widetilde{\beta}_n P_n(s) + \widetilde{\gamma}_n P_{n-1}(s), \qquad \widetilde{\gamma}_n \neq 0,$$

$$\sigma(s)L_{1/q,\omega/q}P_n(s) = \widehat{\alpha}_n P_{n+1}(s) + \widehat{\beta}_n P_n(s) + \widehat{\gamma}_n P_{n-1}(s), \qquad \widehat{\gamma}_n \neq 0,$$

$$P_n(s) = P_n^{[1]}(s) + \delta_n P_{n-1}^{[1]}(s) + \epsilon_n P_{n-2}^{[1]}(s).$$

In particular, in Table 1 we describe these parameters for some families of q-classical orthogonal polynomials.
The first structure relation for the q-semiclassical orthogonal polynomials was established (see [?]), and it reads as follows.
An orthogonal polynomial sequence, $\{B_n\}_{n\geq 0}$, is said to be q-**semiclassical** if

$$\Phi(s)B_n^{[1]}(s) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} B_\nu(s), \; n \geq \sigma, \quad \lambda_{n,n-\sigma} \neq 0, \; n \geq \sigma+1,$$

where Φ is a polynomial of degree t and σ is a non-negative integer such that $\sigma \geq \max\{t-2, 0\}$.
Recently, F. Marcellán and R. Sfaxi [?] have established a second structure relation for the standard semiclassical polynomials which reads as follows

Theorem 1.1. *For any integer $\sigma \geq 0$, any monic polynomial Φ, with $\deg \Phi = t \leq \sigma+2$, and any* SMOP $\{B_n\}_{n\geq 0}$ *with respect to a linear functional u, the following statements are equivalent.*

(i) *There exist an integer $p \geq 1$ and an integer $r \geq \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(x) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(x), \qquad n \geq \max(\sigma, t+1), \tag{3.36}$$

where $B_n^{[1]}(x) = (n+1)^{-1} B'_{n+1}(x)$,

$$\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1, \; n \geq \max(\sigma, t+1), \xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0,$$

$$\langle (\Phi u)', B_n \rangle = 0, \;\; p+1 \leq n \leq 2\sigma+t+1, \;\; \langle (\Phi u)', B_p \rangle \neq 0, \; (\sigma \geq 1),$$

and if $p = t-1$ then $\langle u, B_p^2 \rangle^{-1} \langle u, \Phi B'_p \rangle \notin \mathbb{N}^*$.

(ii) *The linear functional u satisfies*

$$(\Phi u)' + \Psi u = 0,$$

where the pair (Φ, Ψ) is admissible, i.e. the polynomial Φ is monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, *and if $p = t-1$ then $\frac{1}{n!}\Psi^{(n)}(0) \notin -\mathbb{N}^*$, with associated integer σ.*

Now, we are going to extend this result for the q-semiclassical polynomials of the Hahn Tableau. Some years ago, P. Maroni and R. Sfaxi [?] introduced the concept of diagonal sequence for the standard semiclassical polynomials. The following definition extends this definition to the q-semiclassical case.

Definition 1.1. *Let $\{B_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials and ϕ a monic polynomial with* $\deg \phi = t$. *When there exists an integer $\sigma \geq 0$ such that*

$$\phi(s)B_n(s) = \sum_{\nu=n-\sigma}^{n+t} \theta_{n,\nu} B_\nu^{[1]}(s), \;\; \theta_{n,n-\sigma} \neq 0, \quad n \geq \sigma, \tag{3}$$

the sequence $\{B_n\}_{n\geq 0}$ is said to be **diagonal associated with ϕ and index σ.**

(A_1) Big q-Jacobi $\widehat{P}_n(x;a,b,c;q)$ $x \equiv x(s) = q^s$

$$P_n^{[1]}(x;a,b,c;q) = q^{-n}\widehat{P}_n(qx;aq,bq,cq;q)$$

$$\phi(x) = aq(x-1)(bx-c) \qquad \sigma(x) = q^{-1}(x-aq)(x-cq)$$

$$\widehat{\alpha}_n = abq[n] \qquad \widetilde{\alpha}_n = q^{-n}[n]$$

$$\widehat{\beta}_n = -aq[n](1-abq^{n+1})\frac{c+ab^2q^{2n+1}+b(1-cq^n-cq^{n+1}-aq^n(1+q-cq^{n+1}))}{(1-abq^{2n})(1-abq^{2n+2})}$$

$$\widetilde{\beta}_n = q[n](1-abq^{n+1})\frac{c+a^2bq^{2n+1}+a(1-cq^n-cq^{n+1}-bq^n(1+q-cq^{n+1}))}{(1-abq^{2n})(1-abq^{2n+2})}$$

$$\widehat{\gamma}_n = aq[n]\frac{(1-aq^n)(1-bq^n)(1-abq^n)(c-abq^n)(1-cq^n)(1-abq^{n+1})}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$$

$$\widetilde{\gamma}_n = q^n\widehat{\gamma}_n \qquad \delta_n = -\frac{q^n(1-q)}{1-abq^{n+1}}\widehat{\beta}_n \qquad \epsilon_n = abq^{2n}\frac{(1-q^{n-1})(1-q)}{(1-abq^n)(1-abq^{n+1})}\widehat{\gamma}_n$$

(A_2) q-Laguerre $\widehat{L}_n^{(\alpha)}(x;q)$ $x \equiv x(s) = q^s$

$$L_n^{[1]\,(\alpha)}(x;q) = q^{-n}\widehat{L}_n^{(\alpha+1)}(qx;q)$$

$$\phi(x) = ax(x+1) \qquad \sigma(x) = q^{-1}x$$

$$\widehat{\alpha}_n = a[n] \qquad \widehat{\beta}_n = q^{-2n-1}[n](1+q-aq^{n+1}) \qquad \widehat{\gamma}_n = a^{-1}q^{1-4n}[n](1-aq^n)$$

$$\widetilde{\alpha}_n = 0 \qquad \widetilde{\beta}_n = q^{-n}[n] \qquad \widetilde{\gamma}_n = a^{-1}q^{1-3n}(1-aq^n)$$

$$\delta_n = a^{-1}(1-q)\widehat{\beta}_n \qquad \epsilon_n = a^{-1}(1-q^{n-1})(1-q)\widehat{\gamma}_n$$

(A_3) Al-Salam Carlitz I $\widehat{U}_n^{(a)}(x;q)$ $x \equiv x(s) = q^s$

$$U_n^{[1]\,(a)}(x;q) = \widehat{U}_n^{(a)}(x;q)$$

$$\phi(x) = a \quad \sigma(x) = (1-x)(a-x) \quad \widetilde{\alpha}_n = q^{1-n}[n] \quad \widetilde{\beta}_n = q(1+a)[n] \quad \widetilde{\gamma}_n = aq^n[n]$$

(A_4) q-Charlier $\widehat{C}_n(q^{-s};a;q)$

$$C_n^{[1]}(q^{-s};a;q) = \widehat{C}_n(q^{-s};aq^{-1};q)$$

$$\phi(x) = x(x-1) \qquad \sigma(x) = q^{-1}ax$$

$$\widehat{\alpha}_n = [n] \qquad \widehat{\beta}_n = q^{-2n-1}[n](a+aq+q^{n+1}) \qquad \widehat{\gamma}_n = aq^{1-4n}[n](a+q^n)$$

$$\widetilde{\alpha}_n = 0 \qquad \widetilde{\beta}_n = aq^{-n}[n] \qquad \widetilde{\gamma}_n = q^n\widehat{\gamma}_n \qquad \delta_n = (1-q)\widehat{\beta}_n \qquad \epsilon_n = (1-q^{n-1})(1-q)\widehat{\gamma}_n$$

TABLE 1. Some families of q-polynomials of the Hahn Tableau

Obviously, the above finite-type relation, that we will call diagonal relation, is nothing else that an example of second structure relation for such a family. But, some q-semiclassical orthogonal polynomials are not diagonal. As an example, we can mention the case of a q-semiclassical polynomial sequence $\{Q_n\}_{n\geq 0}$ orthogonal with respect to the linear functional v, such that the functional equation: $\Delta^{(1)}v = \Psi v$, with $\deg \Psi = 2$, holds. In fact, the sequence $\{Q_n\}_{n\geq 0}$ satisfies the following relation

$$(x(s+1)+v_{n,0})Q_n(s) = qQ_{n+1}^{[1]}(s) + \rho_n Q_n^{[1]}(s), \; n \geq 0,$$

where the lattice, $x(s)$, is q-linear, i.e. $x(s+1)-qx(s)=\omega$,

$$\begin{aligned}\rho_n = & \frac{q^{n+1}}{\mathfrak{C}}\frac{[n+1]}{\gamma_{n+1}}, \ n\geq 1, \quad \rho_0=0,\\ v_{n,0} = & \frac{\gamma_{n+2}\gamma_{n+1}}{q^n[n+2]}\mathfrak{C}+\rho_n-q\beta_n-\omega, \quad n\geq 0.\end{aligned}$$

Here $\mathfrak{C}$ is a constant, γ_n and β_n are the coefficients of the three-term recurrence relation (TTRR) that the orthogonal polynomial sequence $\{Q_n\}_{n\geq 0}$ satisfies. In fact, this sequence is not diagonal and it will be analyzed more carefully in § 5.1.

The aim of our contribution is to give, under certain conditions, the second structure relation characterizing a q-semiclassical polynomial sequence by a new relation between the sequence of q-polynomials, $\{B_n\}_{n\geq 0}$, and the polynomial sequence of monic q-differences, $\{B_n^{[1]}\}_{n\geq 0}$, as follows

$$\sum_{\nu=n-\sigma}^{n+\sigma}\xi_{n,\nu}B_\nu(s)=\sum_{\nu=n-t}^{n+\sigma}\varsigma_{n,\nu}B_n^{[1]}(s), \quad n\geq \max(t+1,\sigma),$$

where $\xi_{n,n+\sigma}=\varsigma_{n,n+\sigma}=1$, $n\geq\max(t+1,\sigma)$, and there exists $r\geq\sigma+t+1$ such that $\xi_{r,r-\sigma}\varsigma_{r,r-t}\neq 0$.

Notice that when $\sigma=0$ we get the second structure relation (2).

2. Preliminaries and notation

Let u be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space, i.e., the linear space of the linear functionals defined on $\mathbb{P}$. We will denote by $\langle u,f\rangle$ the action of $u\in\mathbb{P}'$ on $f\in\mathbb{P}$ and by $(u)_n:=\langle u,x^n\rangle$, $n\geq 0$, the moments of u with respect to the sequence $\{x^n\}_{n\geq 0}$.

Let us define the following operations in $\mathbb{P}'$. For any polynomial h and any $c\in\mathbb{C}$, let $\Delta^{(1)}u$, hu, and $(x-c)^{-1}u$ be the linear functionals defined on $\mathbb{P}$ by (see [?, ?])

- **(i)** $\langle\Delta^{(1)}u,f\rangle:=-\langle u,\Delta^{(1)}f\rangle, \quad f\in\mathbb{P}$,
- **(ii)** $\langle gu,f\rangle:=\langle u,gf\rangle, \quad f,\ g\in\mathbb{P}$,
- **(iii)** $\langle(x-c)^{-1}u,f\rangle:=\langle u,\theta_c(f)\rangle, \quad f\in\mathbb{P},\ c\in\mathbb{C}, \quad$ where $\theta_c(f)(x)=\frac{f(x)-f(c)}{x-c}$.

Furthermore, for any linear functional u and any polynomial g we get

$$L_{q,\omega}(gu):=\Delta^{(1)}(gu)=g(q^{-1}(x-\omega))\Delta^{(1)}u+\Delta^{(1)}(g(q^{-1}(x-\omega)))u. \tag{4}$$

Let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials (SMP) with $\deg B_n=n$, $n\geq 0$, and $\{u_n\}_{n\geq 0}$ its dual sequence, i.e. $u_n\in\mathbb{P}'$, $n\geq 0$, and $\langle u_n,B_m\rangle:=\delta_{n,m}$, $n,\ m\geq 0$, where $\delta_{n,m}$ is the Kronecker symbol. The next results are very well-known [?].

Lemma 2.1. *For any $u\in\mathbb{P}'$, and any integer $m\geq 1$, the following statements are equivalent.*

- **(i)** $\langle u,B_{m-1}\rangle\neq 0, \quad \langle u,B_n\rangle = 0,\ n\geq m.$
- **(ii)** *There exist $\lambda_\nu\in\mathbb{C}$, $0\leq\nu\leq m-1$, $\lambda_{m-1}\neq 0$, such that $u=\sum_{\nu=0}^{m-1}\lambda_\nu u_\nu$.*

On the other hand, it is straightforward to prove

Lemma 2.2. *For any $(\widehat{t},\widehat{\sigma},\widehat{r})\in\mathbb{N}^3$, $\widehat{r}\geq\widehat{\sigma}+\widehat{t}+1$ and any sequence of monic polynomials $\{\Omega_n\}_{n\geq 0}$, $\deg\Omega_n=n$, $n\geq 0$, with dual sequence $\{w_n\}_{n\geq 0}$ such that*

$$\begin{aligned}\Omega_n(x)= & \sum_{\nu=n-\widehat{t}}^{n}\lambda_{n,\nu}B_\nu(x), \quad n\geq\widehat{t}+\widehat{\sigma}+1, \quad \lambda_{\widehat{r},\widehat{r}-\widehat{t}}\neq 0,\\ \Omega_n(x)= & B_n(x), \quad 0\leq n\leq\widehat{t}+\widehat{\sigma},\end{aligned}$$

we have that $w_k=u_k$ for every $0\leq k\leq\widehat{\sigma}$.

The linear functional u is said to be quasi-definite if, for every non-negative integer, the leading principal Hankel submatrices $H_n = \big((u)_{i+j}\big)_{i,j=0}^{n}$ are non-singular for every $n \geq 0$. Assuming u is quasi-definite, there exists a sequence of monic polynomials $\{B_n\}_{n\geq 0}$ such that (see [?])

- **(i)** $\deg B_n = n,\ n \geq 0,$
- **(ii)** $\langle u, B_n B_m\rangle = r_n \delta_{n,m}$, with $r_n = \ \langle u, B_n^2\rangle \neq 0,\ n \geq 0.$

The sequence $\{B_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials, in short SMOP, with respect to the linear functional u.
If $\{B_n\}_{n\geq 0}$ is a SMOP, with respect to the quasi-definite linear functional u, then it is well-known (see [?]) that its corresponding dual sequence $\{u_n\}_{n\geq 0}$, is

$$u_n = \ r_n^{-1} B_n u,\ n \geq 0. \tag{5}$$

Remark 2.1. *We assume $u_0 = u$, i.e. the linear functional u is normalized.*

On the other hand, (see [?]), the sequence $\{B_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (TTRR)

$$B_{n+1}(x) = (x - \beta_n)B_n(x) - \gamma_n B_{n-1}(x),\ n \geq 0, \tag{6}$$

with $\gamma_n \neq 0,\ n \geq 1$ and $B_{-1}(x) = 0,\ B_0(x) = 1.$
Conversely, given a SMP, $\{B_n\}_{n\geq 0}$, generated by a recurrence relation (6) as above with $\gamma_n \neq 0,\ n \geq 1$, there exists a unique normalized quasi-definite linear functional u such that the family $\{B_n\}_{n\geq 0}$ is the corresponding SMOP. This result is known as Favard Theorem (see [?]).
An important family of linear functionals is constituted by the q-semiclassical linear functionals, i.e., when u is quasi-definite and satisfies

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{7}$$

Here (Φ, Ψ) is an admissible pair of polynomials, i.e., the polynomial Φ is monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, and if $p = t - 1$, then the following condition holds

$$\lim_{q\uparrow 1} \frac{1}{[p]!}[\Delta^{(1)}]^p\, \Psi(0) := \lim_{q\uparrow 1} \frac{1}{[p]!}\overbrace{\Delta^{(1)}\cdots\Delta^{(1)}}^{p}\,\Psi(0) \neq -n, \quad n \in \mathbb{N}^*,$$

where $[m]! = [1][2]\cdots[m]$, $m \in \mathbb{N}^*$, is the q-analog of the usual factorial.
The pair (Φ, Ψ) is not unique. In fact, under certain conditions (7) can be simplified, so we define the class of u as the minimum value of $\max\big(\deg(\Phi) - 2,\ \deg(\Psi) - 1\big)$, for all admissible pairs (Φ, Ψ). The pair (Φ, Ψ) giving the class σ ($\sigma \geq 0$ because $\deg(\Psi) \geq 1$) is unique [?].
When u is q-semiclassical of class σ, the corresponding SMOP is said to be q-semiclassical of class σ.
When $\sigma = 0$, i.e., $\deg \Phi \leq 2$ and $\deg \Psi = 1$, then u is q-classical (Askey-Wilson, q-Racah, Big q-Jacobi, q-Charlier, etc). For more details see [?, ?, ?].

3. Main results

First, we will present particular cases of *diagonal sequences.*
Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be sequences of monic polynomials, $\{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ their corresponding dual sequences. Let ϕ be a monic polynomial of degree t.

Definition 3.1. *The sequence $\{P_n\}_{n\geq 0}$ is said to be compatible with ϕ if $\phi v_n \neq 0$, $n \geq 0$.*

Lemma 3.1. [?, Prop. 2.1] *Let ϕ be as above. For any sequence $\{P_n\}_{n\geq 0}$ compatible with ϕ, the following statements are equivalent.*

- **(i)** *There is an integer $\sigma \geq 0$ such that*

$$\phi(x)Q_n(x) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} P_\nu(x), \quad n \geq \sigma, \tag{8}$$

$$\exists\, r \geq \sigma \ :\quad \lambda_{r,r-\sigma} \neq 0. \tag{9}$$

(ii) *There are an integer $\sigma \geq 0$ and a mapping from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu(m)$ satisfying*

$$\max\{0, m-t\} \leq \mu(m) \leq m+\sigma, \quad m \geq 0, \tag{10}$$

$$\exists\ m_0 \geq 0 \quad \text{with} \quad \mu(m_0) = m_0 + \sigma, \tag{11}$$

such that

$$\phi v_m = \sum_{\nu=m-t}^{\mu(m)} \lambda_{\nu,m} w_\nu, \quad m \geq t, \tag{12}$$
$$\lambda_{\mu(m),m} \neq 0, \quad m \geq 0.$$

Proposition 3.1. [?, Prop. 2.2] *Assume $\{Q_n\}_{n\geq 0}$ is orthogonal and $\{P_n\}_{n\geq 0}$ is compatible with ϕ. Then the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ fulfil the finite-type relations (8)-(9) if and only if there are an integer $\sigma \geq 0$ and a mapping from $\mathbb{N}$ into $\mathbb{N} : m \mapsto \mu(m)$ satisfying (10) and (11). Moreover, there exist $\{k_m\}_{m\geq 0}$ and a sequence $\{\Lambda_{\mu(m)}\}_{m\geq 0}$ of monic polynomials with* $\deg(\Lambda_{\mu(m)}) = \mu(m)$, *$m \geq 0$, such that*

$$\phi v_m = k_m \Lambda_{\mu(m)} w_0, \quad m \geq 0. \tag{13}$$

From these two results we get

Corollary 3.1. [?, Prop. 1.6] *Let ϕ be as above. For sequences of monic orthogonal polynomials (SMOP) $\{P_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ orthogonal with respect to linear functionals v and u, respectively, the following statements are equivalent.*

(i) *There exists an integer $\sigma \geq 0$ such that*

$$\phi(s)P_n(s) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(s), \quad \lambda_{n,n-\sigma} \neq 0,\ n \geq \sigma.$$

(ii) *There exists a monic polynomial sequence $\{\Omega_{n+\sigma}\}_{n\geq 0}$, with* $\deg(\Omega_{n+\sigma}) = n+\sigma$, *$n \geq 0$ and non-zero constants k_n, $n \geq 0$ such that*

$$\phi u_n^{[1]} = k_n \Omega_{n+\sigma} v_0. \tag{14}$$

where $\{u_n^{[1]}\}_{n\geq 0}$ is the dual sequence of $\{B_n^{[1]}\}_{n\geq 0}$.

Thus we can prove

Proposition 3.2. *Any diagonal sequence, $\{B_n\}_{n\geq 0}$, orthogonal with respect a linear functional u is necessarily semiclassical and u satisfies*

$$\Delta^{(1)}(\phi(qx+\omega)\Omega_{n+\sigma}(x)u) = \psi_n(x)u,\ n \geq 0, \tag{15}$$

where

$$\psi_n(s) = \frac{\phi(s+1) - \phi(s-1)}{\Delta x(s)} \Omega_{n+\sigma}(s) - d_n \phi(s)\phi(s-1) B_{n+1}(s), \tag{16}$$

and

$$d_n = [n+1] \frac{\langle u, B_{n+\sigma}^2 \rangle}{\langle u, B_{n+1}^2 \rangle \lambda_{n+\sigma,n}},\ n \geq 0. \tag{17}$$

Furthermore, the sequence $\{\Omega_{n+s}\}_{n\geq 0}$ satisfies

$$\Omega_{n+\sigma}(s)\Delta^{(1)}\Omega_\sigma(s) - \Omega_\sigma(s)\Delta^{(1)}\Omega_{n+\sigma}(s) = \phi(s+1)\{d_n \Omega_\sigma(s) B_{n+1}(s+1) - d_0 \Omega_{n+\sigma}(s) B_1(s+1)\}. \tag{18}$$

Proof: Let $\{B_n\}_{n\geq 0}$ be a diagonal sequence in the sense of Definition 1.1 and assume the linear functional u is normalized. Then from Lemma 3.1 there exist a sequence of monic polynomials $\{\Omega_{n+\sigma}\}_{n\geq 0}$ and non-zero constants $\{k_n\}_{n\geq 0}$ such that

$$\phi u_n^{[1]} = k_n \Omega_{n+\sigma} u.$$

Then

$$\begin{aligned} k_n\Delta^{(1)}(\Omega_{n+\sigma}u) &= \Delta^{(1)}(\phi(q^{-1}(x-\omega)))u_n^{[1]} + \phi(q^{-1}(x-\omega))\Delta^{(1)}u_n^{[1]} \\ &= \Delta^{(1)}(\phi(q^{-1}(x-\omega)))u_n^{[1]} - \frac{[n+1]}{\langle u, B_{n+1}^2\rangle}\,\phi(q^{-1}(x-\omega))B_{n+1}(x)u(s), \end{aligned} \tag{19}$$

as well as

$$\Delta^{(1)}\left(\phi(s)\phi(s-1)\right) = \phi(s)\frac{\phi(s+1)-\phi(s-1)}{\Delta x(s)}. \tag{20}$$

Combining (19) and (20), a straightforward calculation yields (15), (16), and (17).
Taking (15) for $n=0$ and cancelling out $\Delta^{(1)}(\phi(qx+\omega)u)$, from the quasi-definite character of u we obtain (18). □

Corollary 3.2. [?, Corollary 2.3] *If $\{B_n\}_{n\geq 0}$ is a diagonal sequence given by (3), then we get*

$$\frac{1}{2}t \leq \sigma \leq t+2. \tag{21}$$

For a linear functional u, let (Φ,Ψ) be the minimal admissible pair of polynomials with Φ monic, $\deg\Phi = t$, and $\deg\Psi = p \geq 1$, defined as above. To this pair we can associate the non-negative integer $\sigma := \max(t-2, p-1) \geq 0$.
Now, given $\{B_n\}_{n\geq 0}$, a SMOP with respect to u, we get

$$\Phi(s)B_n^{[1]}(s) = \sum_{\nu=0}^{n+t}\lambda_{n,\nu}B_\nu(s), \quad n \geq \max(t-1,0), \tag{22}$$

where $\lambda_{n,n+t} = 1$ and

$$\begin{aligned} \lambda_{n,\nu} &= r_\nu^{-1}\langle u, \Phi(s)B_n^{[1]}(s)B_\nu(s)\rangle = \frac{r_\nu^{-1}}{[n+1]}\langle B_\nu\Phi u, \Delta^{(1)}B_{n+1}\rangle \\ &= -\frac{r_\nu^{-1}}{[n+1]}\langle B_\nu(q^{-1}(x-\omega))\Delta^{(1)}(\Phi u) + \Delta^{(1)}(B_\nu(q^{-1}(x-\omega)))\Phi u, B_{n+1}\rangle, \ 0 \leq \nu \leq n+t. \end{aligned}$$

Lemma 3.2. [?, Prop. 3.2] *For any monic polynomial Φ, $\deg\Phi = t$, and any* SMOP $\{B_n\}_{n\geq 0}$ *with respect to u, the following statements are equivalent.*

(i) *There exists a non-negative integer σ such that*

$$\Phi(s)B_n^{[1]}(s) = \sum_{\nu=n-\sigma}^{n+t}\lambda_{n,\nu}B_\nu(s), \quad n \geq \sigma, \tag{23}$$

$$\lambda_{n,n-\sigma} \neq 0, \quad n \geq \sigma+1. \tag{24}$$

(ii) *There exists a polynomial Ψ, $\deg\Psi = p \geq 1$, such that*

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{25}$$

where the pair (Φ,Ψ) is admissible.

(iii) *There exist a non-negative integer σ and a polynomial Ψ, with $\deg\Psi = p \geq 1$, such that*

$$\Phi(s)\Delta^{(1)}B_n(s-1) + \Psi(s)B_n(s-1) = \sum_{\nu=n-t}^{n+\sigma(n)}\tilde{\lambda}_{n,\nu}B_{\nu+1}(s), \quad n \geq t, \tag{26}$$

$$\tilde{\lambda}_{n,n-t} \neq 0, \quad n \geq t, \tag{27}$$

where $\sigma = \max(p-1, t-2)$, the pair (Φ,Ψ) is admissible, and

$$\sigma(n) = \begin{cases} p-1, & n=0, \\ \sigma, & n \geq 1. \end{cases} \tag{28}$$

We can write

$$\tilde{\lambda}_{n,\nu} = -[\nu+1]\frac{\langle u, B_n^2\rangle}{\langle u, B_{\nu+1}^2\rangle}\lambda_{\nu,n}, \quad 0 \le \nu \le n+\sigma. \tag{29}$$

Proof: (i)⇒ (ii), (iii). Assuming (i), from Lemma 3.1 and taking $P_n = B_n$ and $Q_n = B_n^{[1]}$, we get

$$\Phi u_m = \sum_{\nu=0}^{\mu(m)} \lambda_{\nu,m} u_\nu^{[1]},\ m \ge 0.$$

On the other hand, (24) implies $\mu(m) = m+\sigma$, $m \ge 1$.
Taking into account that

$$\Delta^{(1)} u_m^{[1]} = -[m+1]u_{m+1},\ m \ge 0, \tag{30}$$

we have

$$\Delta^{(1)}(\Phi u_m) = -\sum_{\nu=0}^{\mu(m)} \lambda_{\nu,m}[\nu+1]u_{\nu+1},\ m \ge 0.$$

In accordance with the orthogonality of $\{B_n\}_{n\ge0}$, we get

$$\Delta^{(1)}(\Phi B_m u) = -\Psi_{\mu(m)+1} u,\ m \ge 0, \tag{31}$$

with

$$\Psi_{\mu(m)+1}(s) = \sum_{\nu=0}^{\mu(m)} \lambda_{\nu,m}[\nu+1]B_{\nu+1}(s),\ m \ge 0. \tag{32}$$

Taking $m = 0$ in (31), we have

$$\Delta^{(1)}(\Phi u) = -\Psi_{\mu(0)+1} u. \tag{33}$$

Inserting (33) in (31) and because u is quasi-definite, we get

$$\Phi(s)\Delta^{(1)}B_m(s-1) - \Psi_{\mu(0)+1}(s)B_m(s-1) = -\Psi_{\mu(m)+1}(s),\ m \ge 0.$$

The consideration of the degrees in both hand sides leads to

- If $t-1 > \mu(0)+1$, which implies $t \ge 3$, then $t = \sigma+2$, $\mu(0) < \sigma$.
- If $t-1 \le \mu(0)+1$, then $\mu(0) = \sigma$, $t \le \sigma+2$.

Obviously, the pair $(\Phi, -\Psi_{\mu(0)+1})$ is admissible and putting $p = \mu(0)+1$, we have $\sigma = \max(p-1, t-2)$. So (26) and (27) are valid from (29).
Thus, we have proved that (i)⇒(ii) and (i)⇒(iii).
(ii)⇒(iii). Consider $m \ge 0$. Thus

$$\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1) = \sum_{\nu=0}^{m+\sigma(m)+1} \lambda'_{m,\nu}B_\nu(s).$$

We successively derive from this

$$\langle u, (\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1))B_\mu\rangle = \lambda'_{m,\mu}\langle u, B_\mu^2\rangle,\ 0 \le \mu \le m+\sigma+1.$$

A straightforward calculation yields

$$\langle u, (\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1))B_\mu\rangle = -\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_\mu(s)\rangle. \tag{34}$$

Then

$$-\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_\mu(s)\rangle = \lambda'_{m,\mu}\langle u, B_\mu^2\rangle.$$

Consequently, $\lambda'_{m,\mu} = 0$, $0 \le \mu \le m-t$, $\lambda'_{m,0} = 0$, $m \ge 0$. Moreover, for $\mu = m-t+1$, $m \ge t$,

$$-\langle u, \Phi(s)P_m(s)\Delta^{(1)}P_{m-t+1}(s)\rangle = -[m-t+1]\langle u, B_m^2\rangle = \lambda'_{m,m-t+1}\langle u, B_{m-t+1}^2\rangle.$$

Therefore, for $m \geq t$,

$$\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1) = \sum_{\nu=m-t}^{m+\sigma(m)} \lambda'_{m,\nu+1}B_{\nu+1}(s), \;\; \lambda'_{m,m-t+1} \neq 0.$$

(iii)$\Rightarrow$(i). From (26), we get

$$\begin{aligned}\sum_{\nu=0}^{m+\sigma(m)} \tilde{\lambda}_{m,\nu}\delta_{n,\nu+1} &= \langle u_n, \Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\rangle \\ &= -\langle \Delta^{(1)}(\Phi u_n) - \Psi u_n, B_m(s-1)\rangle.\end{aligned}$$

For $n = 0$, $\langle \Psi u - \Delta^{(1)}(\Phi u), B_m(s-1)\rangle = 0$, $m \geq 0$. Therefore

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{35}$$

Moreover, using (34) and the orthogonality of $\{B_n\}_{n\geq 0}$, we get

$$\langle u_n, \Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\rangle = -r_n^{-1}\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_n(s)\rangle.$$

Furthermore, making $n \to n+1$, we obtain

$$\begin{cases} \langle(\Phi\Delta^{(1)}B_{n+1})u, B_m\rangle = 0, \;\; m \geq n+t+1, \;\; n \geq 0, \\ \langle(\Phi\Delta^{(1)}B_{n+1})u, B_{n+t}\rangle = -r_{n+1}\tilde{\lambda}_{n+t,n} \neq 0, \;\; n \geq 0. \end{cases}$$

According to Lemma 2.1,

$$(\Phi\Delta^{(1)}B_{n+1})u = -\sum_{\nu=n-\sigma}^{n+t} r_n\tilde{\lambda}_{\nu,n}u_\nu, \;\; n \geq \sigma.$$

The orthogonality of $\{B_n\}_{n\geq 0}$ leads to

$$(\Phi\Delta^{(1)}B_{n+1})u = -\sum_{\nu=n-\sigma}^{n+t}\left(\tilde{\lambda}_{\nu,n}\frac{\langle u, B_{n+1}^2\rangle}{\langle u, B_\nu^2\rangle}B_\nu\right)u, \;\; n \geq 0.$$

From (35) and taking into account u is quasi-definite, we finally obtain (23)–(24) in accordance with (29). □

In an analog way we can prove the following result

Lemma 3.3. [**?**, Lemma 3.1] *For any monic polynomial* Φ, $\deg\Phi = t$, *and any* SMOP $\{B_n\}_{n\geq 0}$ *with respect to* u, *the following statements are equivalent.*

(i) *There exists a non-negative integer* σ *such that the polynomials* B_n *satisfy*

$$\Delta^{(1)}(\Phi(s-1)B_n(s)) = \sum_{\nu=n-\sigma-1}^{n+t-1} \lambda_{n,\nu}B_\nu(s), \quad n \geq \sigma+1, \tag{36}$$

$$\lambda_{n,n-\sigma-1} \neq 0, \quad n \geq t+\sigma+2. \tag{37}$$

(ii) *There exists a polynomial* Ψ, $\deg\Psi = p \geq 1$, *such that*

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{38}$$

where the pair (Φ, Ψ) *is admissible.*

(iii) *There exist a non-negative integer* σ *and a polynomial* Ψ, $\deg\Psi = p \geq 1$, *such that*

$$\Phi(s)\Delta^{(1)}B_n(s-1) + \Psi(s)B_n(s-1) - B_n(s)\Delta^{(1)}\Phi(s-1) = \sum_{\nu=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu}B_\nu(s), \quad n \geq t, \tag{39}$$

$$\tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geq t, \tag{40}$$

where $\sigma = \max(p-1, t-2)$ *and the pair* (Φ, Ψ) *is admissible. We can write*

$$\tilde{\lambda}_{n,\nu} = -\frac{\langle u, B_m^2\rangle}{\langle u, B_\nu^2\rangle}\lambda_{\nu,n}, \quad 0 \le \nu \le n+\sigma(n)+1, \; n \ge 0. \tag{41}$$

3.1. **First Characterization of q-semiclassical polynomials.**

Theorem 3.1. *For a monic polynomial* Φ, $\deg \Phi = t$, *and any SMOP* $\{B_n\}_{n\ge 0}$ *with respect to* u, *the following statements are equivalent.*

(i) *There exist a non-negative integer* σ, *an integer* $p \ge 1$, *and an integer* $r \ge \sigma + t + 1$, *with* $\sigma = \max(t-2, p-1)$, *such that*

$$\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+t} v_{n,\nu} B_\nu^{[1]}(s), \quad n \ge \max(\sigma, t), \tag{42}$$

where $\alpha_{n,n+t} = v_{n,n+t} = 1$, $n \ge \max(\sigma, t)$, $\alpha_{r,r-\sigma} v_{r,r-t} \ne 0$,

$$\langle \Delta^{(1)}(\Phi u), B_n\rangle = 0, \; p+1 \le n \le \sigma + 2t + 1, \quad \langle \Delta^{(1)}(\Phi u), B_p\rangle \ne 0,$$

and if $p = t-1$, *then* $\lim_{q\uparrow 1} \langle u, B_p^2\rangle^{-1} \langle u, \Phi\Delta^{(1)} B_p\rangle \ne -m$, $m \in \mathbb{N}^*$.

(ii) *There exists a polynomial* Ψ, $\deg \Psi = p \ge 1$, *such that*

$$\Delta^{(1)}(\Phi u) = \Psi u,$$

and the pair (Φ, Ψ) *is admissible.*

Proof: $(i) \Rightarrow (ii)$. Consider the SMP $\{\Omega_n\}_{n\ge 0}$ defined by

$$\begin{aligned} \Omega_{n+t+1}(s) = & \sum_{\nu=n-t}^{n+t} \frac{[n+t+1]}{[\nu+1]} v_{n,\nu} B_{\nu+1}(s), \quad n \ge \sigma + t + 1, \\ \Omega_n(s) = & \; B_n(s), \quad 0 \le n \le \sigma + 2t + 1. \end{aligned}$$

From (42),

$$\Delta^{(1)}(\Omega_{n+t+1}(s)) = [n+t+1] \sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu} B_\nu(s), \quad n \ge \sigma + t + 1. \tag{43}$$

Since u is quasi-definite, then

$$\begin{aligned} \langle \Delta^{(1)}(\Phi u), \Omega_{n+t+1}\rangle &= -\langle u, \Phi \Delta^{(1)} \Omega_{n+t+1}\rangle \\ &= -[n+t+1] \sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu} \langle u, \Phi B_\nu\rangle = 0, \quad n \ge \sigma + t + 1. \end{aligned}$$

Therefore, $\langle \Delta^{(1)}(\Phi u), \Omega_n\rangle = 0$, $n \ge \sigma + 2t + 1$, and by hypothesis $\langle \Delta^{(1)}(\Phi u), \Omega_n\rangle = 0$, $p+1 \le n \le \sigma + 2t + 1$, then $\langle \Delta^{(1)}(\Phi u), \Omega_n\rangle = 0$ for $n \ge p+1$, and $\langle \Delta^{(1)}(\Phi u), \Omega_p\rangle \ne 0$. Hence, if we denote $\{w_n\}_{n\ge 0}$ the dual sequence of $\{\Omega_n\}_{n\ge 0}$ and apply Lemma 2.1, then

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=1}^{p} \langle \Delta^{(1)}(\Phi u), B_\nu\rangle w_\nu. \tag{44}$$

On the other hand, if we take $\widehat{t} = 2t$, $\widehat{\sigma} = \sigma + 1$, and $\widehat{r} = r + t + 1$, then

$$\begin{aligned} \Omega_n(s) = & \sum_{\nu=n-\widehat{t}}^{n} \tilde{v}_{n,\nu} B_\nu(s), \quad n \ge \widehat{\sigma} + \widehat{t} + 1, \\ \Omega_n(s) = & \; B_n(s), \quad 0 \le n \le \widehat{\sigma} + \widehat{t}, \end{aligned}$$

where

$$\widetilde{v}_{n,\nu} = \frac{[n]}{[\nu]} v_{n-t-1,\nu-1}, \quad n-\widehat{t} \le \nu \le n, \quad n \ge \widehat{\sigma}+\widehat{t}+1,$$
$$\widetilde{v}_{\widehat{r},\widehat{r}-\widehat{t}} = \frac{[r+t+1]}{[r-t+1]} v_{r,r-t} \neq 0, \quad \widehat{r} \ge \sigma+2t+2 = \widehat{\sigma}+\widehat{t}+1.$$

From Lemma 2.2 and (5), it follows that $w_k = u_k = \langle u, B_k^2\rangle^{-1}B_k$, $0 \le k \le \widehat{\sigma} = \sigma+1$. So, relation (44) becomes

$$\Delta^{(1)}(\Phi u) = \Psi u,$$

where

$$\Psi(s) = -\sum_{\nu=1}^{p} \langle u, B_\nu^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_\nu\rangle B_\nu(s),$$

with $\deg\Psi = p$, as well as we have $\langle u, \Phi\Delta^{(1)}B_p\rangle \neq 0$ and, as a consequence, the pair (Φ,Ψ) is admissible with associated integer σ.
$(ii) \Rightarrow (i)$. From Lemma 3.3 (i) and making $n \to n+1$ we have

$$\Delta^{(1)}(\Phi(s-1)B_{n+1}(s)) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n+1,\nu}B_\nu(s), \quad n \ge \sigma, \tag{45}$$

where $\lambda_{n+1,n+t} = [n+t+1]$, $n \ge \sigma$, and $\lambda_{n+1,n-\sigma} \neq 0$, $n \ge t+\sigma+1$.
On the other hand, the orthogonality of $\{B_n\}_{n\ge0}$ yields

$$\Phi(s-1)B_{n+1}(s) = \sum_{\nu=n-t}^{n+t} \frac{\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s)\rangle}{\langle u, B_{\nu+1}^2\rangle} B_{\nu+1}(s), \quad n \ge t-1.$$

Hence,

$$\Delta^{(1)}(\Phi(s-1)B_{n+1}(s)) = \sum_{\nu=n-t}^{n+t} \frac{[\nu+1]\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s)\rangle}{\langle u, B_{\nu+1}^2\rangle} B_\nu^{[1]}(s), \quad n \ge t. \tag{46}$$

From (45) and (46), we obtain (42) with

$$\begin{aligned} \alpha_{n,\nu} &= \frac{\lambda_{n+1,\nu}}{[n+t+1]}, \quad n-\sigma \le \nu \le n+t, \\ v_{n,\nu} &= \frac{[\nu+1]\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s)\rangle}{[n+t+1]\langle u, B_{\nu+1}^2\rangle}, \quad n-t \le \nu \le n+t, \\ \alpha_{n,n-\sigma}v_{n,n-t} &\neq 0, \quad n \ge \sigma+t+1. \end{aligned}$$

Then,

$$\langle \Delta^{(1)}(\Phi u), B_n\rangle = -\langle u, \Phi\Delta^{(1)}B_n\rangle = \begin{cases} 0, \ p+1 \le n \le \sigma+2t+1, \\ \dfrac{1}{[p]!}[\Delta^{(1)}]^p\Psi(0)\langle u, B_p^2\rangle, \ n = p = \deg\Psi, \end{cases}$$

and if $p = t-1$, the q-admissibility of (Φ,Ψ) yields $\lim\limits_{q\uparrow1}\langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_p\rangle \neq -m$, $m \in \mathbb{N}^*$.
□

In the case of q-classical linear functionals, we get the following result

Corollary 3.3. *Let $\{B_n\}_{n\ge0}$ be a SMOP with respect to u and a monic polynomial Φ, with $\deg\Phi = t \le 2$, such that $\langle u, \Phi\rangle \neq 0$, then the following statements are equivalent.*

- **(i)** *The linear functional u is q-classical, i.e. there exists a polynomial Ψ with $\deg\Psi = 1$ such that $\Delta^{(1)}(\Phi u) = \Psi u$.*
- **(ii)** *$\sum_{\nu=n}^{n+t}\alpha_{n,\nu}B_\nu(s) = \sum_{\nu=n-t}^{n+t} v_{n,\nu}B_\nu^{[1]}(s)$, $n \ge t$. Furthermore, there exists an integer $r \ge t+1$ such that $\alpha_{r,r}v_{r,r-t} \neq 0$, and if $t = 2$ then $\lim_{q\uparrow1}\langle u, B_1^2\rangle^{-1}\langle u, \Phi\rangle \neq -m$, $m \in \mathbb{N}^*$.*

3.2. **Second Characterization of q-semiclassical polynomials.** From the previous characterization, we can not recover the second structure relation of q-classical orthogonal polynomials (2). Our goal is to establish the characterization that allows us to deduce such a case.
First, we have the following result.

Proposition 3.3. *For any monic polynomial Φ, with $\deg\Phi = t$, and any SMOP $\{B_n\}_{n\geq 0}$ with respect to u, the following statements are equivalent.*

(i) *There exists a polynomial Ψ, $\deg\Psi = p \geq 1$, such that*

$$\Delta^{(1)}(\Phi u) = \Psi u, \tag{47}$$

where the pair (Φ, Ψ) is admissible.

(ii) *There exist a non-negative integer σ and a polynomial Ψ, with $\deg\Psi = p \geq 1$, such that*
(48)

$$\Phi(s)[\Delta^{(1)}]^2 B_n(s-1) + \Delta^{(1)}(\Psi(s)B_n(s-1)) - B_n(s)[\Delta^{(1)}]^2\Phi(s-1) = \sum_{\nu=n-\sigma}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s),\ n\geq\sigma,$$

where $\vartheta_{n,n-\sigma}\neq 0$ either $n\geq\sigma+t+1$ or $n=\sigma+t$ and $p\geq t-1$, $\sigma=\max(t-2,p-1)$, and the pair (Φ,Ψ) is admissible. We can write

$$\vartheta_{n,\nu} = \frac{\langle u, B_n^2\rangle}{\langle u, B_\nu^2\rangle}\vartheta_{\nu,n}, \quad 0\leq\nu\leq n+\sigma(n),\ n\geq 0. \tag{49}$$

Proof: We have
(50)

$$\Phi(s)[\Delta^{(1)}]^2 B_n(s-1) + \Delta^{(1)}(\Psi(s)B_n(s-1)) - B_n(s)[\Delta^{(1)}]^2\Phi(s-1) = \sum_{\nu=0}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s),\ n\geq 0,$$

where for all integers $0\leq\nu\leq n+\sigma(n)$, and $n\geq 0$,

$$\langle u, B_\nu^2\rangle\vartheta_{n,\nu} = \langle u, (\Phi(s)[\Delta^{(1)}]^2 B_n(s-1) + \Delta^{(1)}(\Psi(s)B_n(s-1)) - B_n(s)[\Delta^{(1)}]^2\Phi(s-1))B_\nu\rangle.$$

Taking into account (5) and (48), a straightforward calculation leads to

$$\langle u, B_\nu^2\rangle\vartheta_{n,\nu} = \langle u, (\Phi(s)[\Delta^{(1)}]^2 B_\nu(s-1) + \Delta^{(1)}(\Psi(s)B_\nu(s-1)) - B_\nu(s)[\Delta^{(1)}]^2\Phi(s-1))B_n\rangle.$$

Therefore, inserting (50)

$$\langle u, B_\nu^2\rangle\vartheta_{n,\nu} = \sum_{i=0}^{\nu+\sigma(\nu)} \vartheta_{\nu,i}\langle u, B_n^2\rangle\delta_{i,n} = \vartheta_{\nu,n}\langle u, B_n^2\rangle.$$

In particular, for $0\leq\nu\leq n-\sigma-1$, then $n\geq\nu+\sigma+1\geq\nu+\sigma(\nu)+1$. Thus, we deduce $\vartheta_{\nu,n}=0$. Hence $\vartheta_{n,\nu}=0$, for $0\leq\nu\leq n-\sigma-1$.
For $\nu=n-\sigma$ and $n\geq\sigma+t$, we obtain

$$\begin{aligned}\langle u, B_{n-\sigma}^2\rangle\vartheta_{n,n-\sigma} = {} & \langle u, \Delta^{(1)}\big(\Phi(s)\Delta^{(1)}B_{n-\sigma}(s-1) + \Psi(s)B_{n-\sigma}(s-1)\big)\rangle \\ & -\langle u, \Delta^{(1)}\big(B_{n-\sigma}(s)\Delta^{(1)}\Phi(s-1)\big)B_n\rangle = \sum_{nu=0}^{n+1}\tilde\lambda_{n-\sigma,\nu}\langle u, B_n\Delta^{(1)}B_\nu\rangle \\ = {} & [n+1]\tilde\lambda_{n-\sigma,n+1}\langle u, B_n^2\rangle.\end{aligned}$$

But, from (40), we get $\vartheta_{n,n-\sigma}\neq 0$, either $n\geq\sigma+t+1$, or $n=\sigma+t$ and $p\geq t-1$.
As a consequence,

$$\Phi(s)[\Delta^{(1)}]^2 B_n(s-1) + \Delta^{(1)}(\Psi(s)B_n(s-1)) - B_n(s)[\Delta^{(1)}]^2\Phi(s-1) = \sum_{\nu=n-\sigma}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s),\ n\geq\sigma.$$

(ii)⇒(i). From (48)

$$\langle \Delta^{(1)}(\Phi(s-1)\Delta^{(1)}u) + \big((\Delta^{(1)}\Phi(s-1)) - \Psi(s)\big)\Delta^{(1)}u, B_n(s-1)\rangle = 0, \quad n \geq \sigma+1,$$
$$\langle \Delta^{(1)}(\Phi(s-1)\Delta^{(1)}u) + \big((\Delta^{(1)}\Phi(s-1)) - \Psi(s)\big)\Delta^{(1)}u, B_n(s-1)\rangle = \langle u,1\rangle \vartheta_{n,0}, \quad n \leq \sigma.$$

According to Lemma 2.1

$$\begin{aligned}\Delta^{(1)}(\Phi(s-1)\Delta^{(1)}u) + \big((\Delta^{(1)}\Phi(s-1)) - \Psi(s)\big)\Delta^{(1)}u =& \sum_{n=0}^{\sigma} \frac{\langle u,1\rangle\vartheta_{n,0}}{\langle u, B_n^2\rangle} B_n(\nabla u - u)\\ =& \sum_{n=0}^{\sigma(0)} \vartheta_{0,n} B_n(\nabla u - u).\end{aligned}$$

Finally, a direct calculation yields

$$\Delta^{(1)}\big(\Delta^{(1)}(\Phi u) - \Psi u\big) = 0,$$

then $\Delta^{(1)}(\Phi u) - \Psi u = 0$.
Moreover, since $\sigma(n) = \sigma$ and $\vartheta_{n,n+\sigma} = [n+\sigma+1]\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, for $n \geq t+1$, then $\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, $n \geq t+1$. The q-admissibility of the pair (Φ, Ψ) follows taking into account the value of $\tilde{\lambda}_{n+\sigma(n)+1}$.
□

Our main result is the next one.

Theorem 3.2. *For any monic polynomial Φ, $\deg \Phi = t$, and any SMOP $\{B_n\}_{n\geq 0}$ with respect to u, the following statements are equivalent.*

(i) *There exist a non-negative integer σ, an integer $p \geq 1$, and an integer $r \geq \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

(51)
$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(s),$$

where $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geq \max(\sigma, t+1)$, $\xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0$,

$$\begin{cases} \langle \Delta^{(1)}(\Phi u), B_m\rangle = 0, \quad p+1 \leq m \leq 2\sigma+t+1, \\ \langle \Delta^{(1)}(\Phi u), B_p\rangle \neq 0, \end{cases}$$

and if $p = t-1$, then $\lim_{q\uparrow 1} \langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_p\rangle \neq m$, $m \in \mathbb{N}^*$ (q-admissibility condition).

(ii) *There exists a polynomial Ψ, $\deg \Psi = p \geq 1$, such that*

(52)
$$\Delta^{(1)}(\Phi u) = \Psi u,$$

where the pair (Φ, Ψ) is admissible.

Proof: (i)⇒ (ii). Let us consider the SMP $\{\Xi_n\}_{n\geq 0}$ given by

$$\begin{aligned}\Xi_{n+\sigma+1}(x) =& \sum_{\nu=n-t}^{n+\sigma} \frac{[n+\sigma+1]}{[\nu+1]} \varsigma_{n,\nu} B_{\nu+1}(x), \quad n \geq \sigma+t+1,\\ \Xi_n(x) =& B_n(x), \quad 0 \leq n \leq 2\sigma+t+1.\end{aligned}$$

A direct calculation yields

$$\Delta^{(1)}\Xi_{n+\sigma+1}(s) = [n+\sigma+1] \sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s), \quad n \geq \sigma+t+1.$$

Taking into account the linear functional u is quasi-definite, we get

$$\langle \Delta^{(1)}(\Phi u), \Xi_{n+\sigma+1}\rangle = -\langle u, \Phi\Delta^{(1)}\Xi_{n+\sigma+1}(s)\rangle = -[n+\sigma+1] \sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu}\langle u, \Phi B_\nu\rangle = 0, \quad n \geq \sigma+t+1.$$

From the assumption and Lemma 2.1, if we denote $\{w_n\}_{n\geq 0}$ the dual sequence of $\{\Xi_n\}_{n\geq 0}$, then we get

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=0}^{p} \langle \Delta^{(1)}(\Phi u), B_\nu \rangle w_k. \tag{53}$$

Taking $\widehat{t} = \sigma + t$, $\widehat{\sigma} = \sigma + 1$, and $\widehat{r} = r + \sigma + 1$, the polynomials Ξ_n can be rewritten as follows

$$\begin{aligned} \Xi_n(x) = & \sum_{\nu=n-\widehat{t}}^{n} \widetilde{\varsigma}_{n,\nu} B_v(x), \quad n \geq \widehat{\sigma} + \widehat{t} + 1, \\ \Xi_n(x) = & \ B_n(x), \quad 0 \leq n \leq \widehat{\sigma} + \widehat{t}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{\varsigma}_{n,\nu} = & \ \frac{[n]}{[\nu]} \varsigma_{n-\sigma-1,\nu-1}, \quad n - \widehat{t} \leq \nu \leq n, \ n \geq \sigma + \widehat{t} + 1, \\ \widetilde{\varsigma}_{\widehat{r},\widehat{r}-\widehat{t}} = & \ \frac{[r+\sigma+1]}{[r-t+1]} \varsigma_{r,r-t} \neq 0, \quad \widehat{r} \geq 2\sigma + t + 2 \geq \widehat{\sigma} + \widehat{t} + 1. \end{aligned}$$

From Lemma 2.2, $w_k = u_k = \langle u, B_k^2 \rangle^{-1} B_k u$, $0 \leq k \leq \widehat{\sigma} = \sigma + 1$. So, (53) becomes

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=1}^{p} \left(\frac{\langle \Delta^{(1)}(\Phi u), B_\nu \rangle}{\langle u, B_\nu^2 \rangle} B_\nu \right) u = \Psi u.$$

Since $\langle \Delta^{(1)}(\Phi u), B_p \rangle \neq 0$, then $\deg \Psi = p$.
From the assumption, if $p = t - 1$, then

$$\lim_{q\uparrow 1} \frac{1}{[p]!} \left[\Delta^{(1)}\right]^p \Psi(0) = \lim_{q\uparrow 1} \frac{\langle \Delta^{(1)}(\Phi u), B_p \rangle}{\langle u, B_p^2 \rangle} = -\lim_{q\uparrow 1} \frac{\langle u, \Phi \Delta^{(1)} B_p \rangle}{\langle u, B_p^2 \rangle} \neq -m, \quad m \in \mathbb{N}^*.$$

Hence, the pair (Φ, Ψ) is admissible with associated integer σ.
(ii)⇒(i). From Lemma 3.2(iii), there exists a polynomial Ψ, $\deg \Psi = p \geq 1$, such that

$$\Phi(s-1)\Delta^{(1)} B_n(s-1) + \Psi(s) B_n(s-1) - B_n(s)\Delta^{(1)}\Phi(s-1) = \sum_{\nu=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu} B_\nu(s), \quad n \geq t, \tag{54}$$

where $\tilde{\lambda}_{n,n-t+1} \neq 0$, $n \geq t$, $\sigma = \max(t-2, p-1)$, and the pair (Φ, Ψ) is admissible.
Taking q-differences in both hand sides of (54), we get
(55)

$$\Phi(s)[\Delta^{(1)}]^2 B_n(s-1) + \Delta^{(1)}(\Psi(s) B_n(s-1)) - B_n(s)[\Delta^{(1)}]^2 \Phi(s-1) = \sum_{\nu=n-t}^{n+\sigma(n)} \zeta_{n,\nu} B_\nu^{[1]}(s), \ n \geq t,$$

where $\zeta_{n,\nu} = [\nu+1]\tilde{\lambda}_{n,\nu+1}$, $0 \leq \nu \leq n + \sigma(n)$, $n \geq t$.
From (48) and (55), we obtain (51) where

$$\begin{aligned} \xi_{n,\nu} = & \ \frac{\vartheta_{n,\nu}}{\vartheta_{n,n+\sigma}}, \quad n - \sigma \leq \nu \leq n + \sigma, \\ \varsigma_{n,\nu} = & \ \frac{[\nu+1]\tilde{\lambda}_{n,\nu+1}}{\vartheta_{n,n+\sigma}}, \quad n - t \leq \nu \leq n + t, \\ \xi_{n,n-\sigma}\varsigma_{n,n-t} = & \ \frac{[n-t+1]}{\vartheta_{n,n+\sigma}^2} \vartheta_{n,n-\sigma} \tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geq \sigma + t + 1. \end{aligned}$$

Finally,

$$\langle \Delta^{(1)}(\Phi u), B_n \rangle = \langle u, \Psi B_n \rangle = \begin{cases} 0, & p + 1 \leq n \leq 2\sigma + t + 1, \\ (\langle u, B_p^2 \rangle / [p]!)[\Delta^{(1)}]^p \Psi(0) \neq 0, & n = p = \deg \Psi. \end{cases}$$

From the admissibility of the pair (Φ, Ψ), if $p = t - 1$, then $\langle u, B_p^2 \rangle^{-1} \langle u, \Phi \Delta^{(1)} B_p \rangle \neq m$, $m \in \mathbb{N}^*$.
□

4. The uniform lattice $x(s) = s$

As a direct consequence from the operator $L_{q,\omega}$ and the q-linear lattice $x(s)$, we can recover the uniform lattice setting $x(s) = (q^s - 1)/(q-1)$ and taking limit $q \to 1$. For instance, for Δ-classical orthogonal polynomials the structure relations (1) and (2) have been studied in [?].

Theorem 4.1. First Characterization of discrete semiclassical polynomials *For a monic polynomial Φ, $\deg \Phi = t$, and any SMOP $\{B_n\}_{n\geq 0}$ with respect to u, the following statements are equivalent.*

(i) *There exist a non-negative integer σ, an integer $p \geq 1$, and an integer $r \geq \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+t} v_{n,\nu} B_\nu^{[1]}(s), \quad n \geq \max(\sigma, t), \tag{56}$$

where $B_n^{[1]}(s) := (n+1)^{-1} \Delta B_{n+1}(s)$, $\alpha_{n,n+t} = v_{n,n+t} = 1$, $n \geq \max(\sigma, t)$, $\alpha_{r,r-\sigma} v_{r,r-t} \neq 0$,

$$\langle \Delta(\Phi u), B_n \rangle = 0, \ p+1 \leq n \leq \sigma + 2t + 1, \quad \langle \Delta(\Phi u), B_p \rangle \neq 0,$$

and if $p = t-1$, then $\langle u, B_p^2 \rangle^{-1} \langle u, \Phi \Delta B_p \rangle \neq -m$, $m \in \mathbb{N}^*$.

(ii) *There exists a polynomial Ψ, $\deg \Psi = p \geq 1$, such that*

$$\Delta(\Phi u) = \Psi u,$$

and the pair (Φ, Ψ) is admissible.

Theorem 4.2. Second Characterization of discrete semiclassical polynomials *For any monic polynomial Φ, $\deg \Phi = t$, and any SMOP $\{B_n\}_{n\geq 0}$ with respect to u, the following statements are equivalent.*

(i) *There exist a non-negative integer σ, an integer $p \geq 1$, and an integer $r \geq \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(s), \tag{57}$$

where $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geq \max(\sigma, t+1)$, $\xi_{r,r-\sigma} \varsigma_{r,r-t} \neq 0$,

$$\begin{cases} \langle \Delta(\Phi u), B_m \rangle = 0, & p+1 \leq m \leq 2\sigma + t + 1, \\ \langle \Delta(\Phi u), B_p \rangle \neq 0, \end{cases}$$

and if $p = t-1$, then $\langle u, B_p^2 \rangle^{-1} \langle u, \Phi \Delta B_p \rangle \neq m$, $m \in \mathbb{N}^*$ (admissibility condition).

(ii) *There exists a polynomial Ψ, $\deg \Psi = p \geq 1$, such that*

$$\Delta(\Phi u) = \Psi u, \tag{58}$$

where the pair (Φ, Ψ) is admissible.

The proofs are analogous to the original ones setting $\omega = 1$, and taking limit $q \uparrow 1$. Therefore $L_{q,1} \equiv \Delta^{(1)}$ becomes Δ and $[n]$ becomes n.

Remark 4.1. *Δ-semiclassical linear functionals have been studied in* [?].

5. Examples

5.1. **First example.** Let $\{Q_n\}_{n\geq 0}$ be a SMOP that satisfies the following relation

$$(x(s+1) + v_{n,0}) Q_n(s) = q Q_{n+1}^{[1]}(s) + \rho_n(s) Q_n^{[1]}(s), \tag{59}$$

where the lattice, $x(s)$, is q-linear, i.e. $x(s+1)-qx(s)=\omega$,

$$\begin{aligned}\rho_n &= \frac{q^{n+1}}{\mathfrak{C}}\frac{[n+1]}{\gamma_{n+1}}, \; n\geq 1, \quad \rho_0=0,\\ \upsilon_{n,0} &= \frac{\gamma_{n+2}\gamma_{n+1}}{q^n[n+2]}\mathfrak{C}+\rho_n-q\beta_n-\omega, \quad n\geq 0,\end{aligned}$$

and $\mathfrak{C}$ is a constant, being $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ the coefficients of the TTRR

$$xQ_n = Q_{n+1}+\beta_n Q_n+\gamma_n Q_{n-1}, \quad n\geq 1.$$

Then, from the above TTRR and Theorem 3.1, we get $\{Q_n\}_{n\geq 0}$ is a sequence of q-semiclassical orthogonal polynomials with respect to the linear functional v, solution of the Pearson equation

$$\Delta^{(1)}v=\Psi v, \tag{60}$$

of class $\sigma=1$, with $\Phi(x)=1$ and $\deg\Psi=2$.
Then, it also satisfies the following relation

$$Q_n^{[1]}(s)=Q_n(s)+\lambda_{n,n-1}Q_{n-1}(s), \tag{61}$$

where $\lambda_{n,n-1}=\dfrac{\gamma_{n+1}\gamma_n}{q^n[n+1]}\mathfrak{C}$.
In fact, a straightforward calculation gives $\Psi(x)=-\dfrac{\mathfrak{C}}{q}\,Q_2(x)-\dfrac{1}{\gamma_1}\,Q_1(x)$.

Lemma 5.1. *Let $\{Q_n\}_{n\geq 0}$ be a SMOP with respect to the linear functional v satisfying (60). Then the sequence $\{Q_n\}_{n\geq 0}$ is not diagonal.*

Proof: Assume $\{Q_n\}_{n\geq 0}$ is diagonal with respect to ϕ, with $\deg\phi=t$, and index σ. Then from Corollary 3.2, $t/2\leq\sigma\leq t+2$ and we have the following diagonal relation

$$\phi(s)Q_n(s)=\sum_{\nu=n-\sigma}^{n+t}\theta_{n,\nu}Q_\nu^{[1]}(s), \quad \theta_{n,n-\sigma}\neq 0, \; n\geq\sigma.$$

If we denote by $\{v_n\}_{n\geq 0}$ and $\{v_n^{[1]}\}_{n\geq 0}$ the dual sequences of $\{Q_n\}_{n\geq 0}$ and $\{Q_n^{[1]}\}_{n\geq 0}$, respectively, then by Proposition 3.1 the last relation is equivalent to

$$\phi v_n^{[1]}=k_n\Omega_{n+\sigma}v, \; n\geq 0, \tag{62}$$

where $k_n=\langle v,Q_{n+\sigma}^2\rangle^{-1}\theta_{n+\sigma,n}$, and

$$\Omega_{n+\sigma}(x)=\sum_{\nu=0}^{n+\sigma}\frac{\theta_{\nu,n}}{\theta_{n+\sigma,n}}\frac{\langle v,Q_{n+\sigma}^2\rangle}{\langle v,Q_\nu^2\rangle}Q_\nu(x), \quad n\geq 0.$$

It is clear that v satisfies an infinite number of relations as (62). Indeed, by multiplying both hand sides of (62) by a monic polynomial, we get another diagonal relation.
For this reason, we will assume $t=\deg\phi$ is the minimum non-negative integer such that v satisfies diagonal relations as (62), i.e. the Eq. (62) cannot be simplified.
Notice that $t\geq 1$. Indeed, if we suppose that $t=0$, then $0\leq\sigma\leq 2$ and we recover the first structure relation characterizing q-classical sequences. This contradicts the fact that the sequence $\{Q_n\}_{n\geq 0}$ is q-semiclassical of class one.
Consequently, since $t\geq 1$ then $\sigma\geq 1$. Taking q-differences in both hand sides of (62) and using (5), from (60) and $\Delta^{(1)}v_n^{[1]}=-[n+1]v_{n+1}$, we obtain

$$\widetilde{\phi}v_n^{[1]}=k_n\psi_n v, \quad n\geq 0, \tag{63}$$

where

$$\begin{aligned}\widetilde{\phi}(s) =& \ [t]^{-1}\Delta^{(1)}\phi(s),\\ \psi_n(s) =& \ [t]^{-1}\big(\Omega_{n+\sigma}(s+1)\Psi(s)+\Delta^{(1)}\Omega_{n+\sigma}(s)+d_n\phi(s+1)Q_{n+1}(s)\big),\ n\geq 0,\\ d_n =& \ [n+1](\langle v, Q_{n+1}^2\rangle k_n)^{-1},\ n\geq 0.\end{aligned}$$

Notice that the polynomial $\widetilde{\phi}$ is monic with $\deg\widetilde{\phi}=t-1$.
Moreover, taking into account u is a quasi-definite linear functional, combining (62) and (63) we obtain $\widetilde{\phi}(x)\Omega_{n+\sigma}(x)=\phi(x)\psi_n(x)$, and analyzing the highest degree of this relation, we get ψ_n is a monic polynomial with $\deg\psi_n=n+\sigma-1$. But, this contradicts the fact that $t=\deg\phi$ is the minimum nonnegative integer such that v satisfies diagonal relations as (62). □

5.2. **The q-Freud type polynomials.** Let $\{P_n\}_{n\geq 0}$ be a SMOP with respect to a linear functional u such that $(u)_0=\langle u,1\rangle=1$ and the following relation

$$\Delta^{(1)}P_n(s)=[n]P_{n-1}(s)+a_nP_{n-3}(s),\quad n\geq 2, \tag{64}$$

holds, where $P_{-1}\equiv 0$, $P_0\equiv 1$, and $P_1(x)=x$, being $x\equiv x(s)=q^s$, i.e. $\omega=0$.
We know that this family satisfies a TTRR, i.e. there exist two sequences of complex numbers $\{b_n\}_n$ and $\{c_n\}_n$, $c_n\neq 0$, such that

$$xP_n=P_{n+1}+b_nP_n+c_nP_{n-1},\quad n\geq 1.$$

Furthermore, from a direct calculation we get $a_n=K(q)q^{-n}c_nc_{n-1}c_{n-2}$, $n\geq 2$. In fact, the parameters c_n satisfy the non-linear recurrence relation

$$q[n]c_{n-1}+K(q)q^{-n+1}c_nc_{n-1}c_{n-2}=[n-1]c_n+K(q)q^{-n-1}c_{n+1}c_nc_{n-1},\quad n\geq 1,$$

with $c_0=0$, $c_1=-P_2(0)\neq 0$, and $\lim_{q\uparrow 1}K(q)=4$.
Moreover, from Proposition 3.2 we deduce that $\Phi\equiv 1$ and thus $\sigma=2$. As a consequence, Ψ is a polynomial of degree 3. In other words, u is a q-semiclassical linear functional of class 2, i.e. u satisfies the following distributional equation

$$\Delta^{(1)}u=\Psi u,\quad \deg\Psi=3. \tag{65}$$

Lemma 5.2. $\Psi(x)=-K(q)q^{-3}P_3(x)-c_1^{-1}P_1(x)$.

So, (65) is the q-analog of the Pearson equation for the Freud case.
Proof: From our hypothesis Ψ is a polynomial of degree 3, so $\Psi(x)=e_0P_0+e_1P_1+e_2P_2+e_3P_3$. Then, taking into account $d_n^2=c_nd_{n-1}^2$, $n\geq 1$, and the value of a_n, $n\geq 3$, we get

$$\begin{aligned}&e_0d_0^2=e_0\langle u,P_0^2\rangle=\langle\Psi u,P_0\rangle=-\langle u,\Delta^{(1)}P_0\rangle=0,\\ &e_1d_1^2=e_1\langle u,P_1^2\rangle=\langle\Psi u,P_1\rangle=-\langle u,\Delta^{(1)}P_1\rangle=-1,\\ &e_2d_2^2=e_2\langle u,P_2^2\rangle=\langle\Psi u,P_2\rangle=-\langle u,\Delta^{(1)}P_2\rangle\overset{(64)}{=}-\langle u,[2]P_1\rangle=0,\\ &e_3d_3^2=e_3\langle u,P_3^2\rangle=\langle\Psi u,P_3\rangle=-\langle u,\Delta^{(1)}P_3\rangle\overset{(64)}{=}-\langle u,[3]P_2+a_3P_0\rangle=-a_3.\end{aligned}$$

□

From Theorem 3.2, we can write the second structure relation as follows

$$B_{n+2}+\xi_{n,n+1}B_{n+1}+\xi_{n,n}B_n+\xi_{n,n-1}B_{n-1}+\xi_{n,n-2}B_{n-2}=B_{n+2}^{[1]}+\varsigma_{n,n+1}B_{n+1}^{[1]}+\varsigma_{n,n}B_n^{[1]}. \tag{66}$$

Using (64) we get

$$\begin{aligned}&\xi_{n,n+1}=\varsigma_{n,n+1}, && \xi_{n,n}=[n+3]^{-1}a_{n+3}+\varsigma_{n,n},\\ &\xi_{n,n-1}=[n+2]^{-1}\varsigma_{n,n+1}a_{n+2}, && \xi_{n,n-2}=[n+1]^{-1}\varsigma_{n,n}a_{n+1}.\end{aligned}$$

Moreover, combining both structure relations if $P_n(x) = \sum_{j=0}^{n} \lambda_{n,j}\, x^{n-j}$, then $\lambda_{n,2k+1} = 0$ for nonnegative integers n, k such that $0 \le k \le (n-1)/2$, and

$$\lambda_{n,0} = 1, \quad \lambda_{n,2k+2} = \frac{[n]c_{n-1}\lambda_{n-2,2k} + a_n\lambda_{n-3,2k}}{[n-2k-2]-[n]}, \quad 1 \le k \le n/2.$$

In fact, with these values, we obtain $c_n = \lambda_{n,2} - \lambda_{n+1,2}$, $b_n = \lambda_{n,1} - \lambda_{n+1,1} = 0$, and $\xi_{n,n+1} = \xi_{n,n-1} = \varsigma_{n,n+1} = 0$, $n \ge 0$. Hence, we can rewrite (66) as

$$(x^2 + \widetilde{v}_{n,0})B_n = B_{n+2}^{[1]} + \widetilde{\rho}_n B_n^{[1]}, \tag{67}$$

where $\widetilde{v}_{n,0} = \dfrac{a_{n+3}}{[n+3]} + \dfrac{q^{n+1}[n+1]}{K(q)c_{n+1}} - c_{n+1} - c_n$, and $\widetilde{\rho}_n = \dfrac{q^{n+1}[n+1]}{K(q)c_{n+1}}$.

Lemma 5.3. *The moments of the linear functional u, $\{(u)_n\}_{n\ge0}$, satisfy the following relation*

$$[n+1](u)_n = K(q)q^{-3}(u)_{n+4} + \left(\frac{1}{c_1} - \frac{[3]c_2 + a_3}{q(1+q)}\right)(u)_{n+2}, \quad n \ge 0, \tag{68}$$

where $(u)_0 = 1$.

Therefore, taking into account that $(u)_1 = (u)_3 = 0$, we can deduce u is a symmetric linear functional, i.e. $(u)_{2n+1} = \langle u, x^{2n+1}\rangle = 0$, $n \ge 0$.

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