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Polinomios ortogonales Ortogonalidad y dualidad

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Classical orthogonal polynomials

Given a linear functional $\mathbf{u} : \mathbb{P}[\mathbf{x}] \to \mathbb{C}$ fulfilling the Pearson type equation

(PE) $\mathscr{D}\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}, \quad \deg \phi \le 2, \deg \psi = 1$ $\to p_n(x) = \text{ops}(\mathbf{u})$

(TTRR) $x p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x)$ $\gamma_n = \alpha_{n-1}\frac{\langle \mathbf{u}, p_n^2(x)\rangle}{\langle \mathbf{u}, p_{n-1}^2(x)\rangle}$

(1RE) $\phi(x)(\mathscr{D}p_n)(x) = \widetilde{\alpha}_n p_{n+1}(x) + \widetilde{\beta}_n p_n(x) + \widetilde{\gamma}_n p_{n-1}(x)$ $\widetilde{\gamma}_n \neq 0$

(2RE) $p_n(x) = \widehat{\alpha}_n(\mathscr{D}p_{n+1})(x) + \widehat{\beta}_n(\mathscr{D}p_n)(x) + \widehat{\gamma}_n(\mathscr{D}p_{n-1})(x)$ $\widehat{\gamma}_n \neq \gamma_n$

(SODE) $\phi(x)\mathscr{D}\mathscr{D}^* p_n(x) + \psi(x)\mathscr{D}p_n(x) + \lambda_n p_n(x) = 0$ $\lambda_n = -n\psi' - \frac{n(n-1)}{2}\phi''(x)$

where $\mathscr{D} := \frac{d}{dx}$ in the cont. case, $\mathscr{D} := \Delta$ in the disc. case, $\mathscr{D} := \mathscr{D}_q$ in the q-disc. case.

Laguerre-type cases

These are the families for which $\widetilde{\gamma}_n = 0$.

In such a case, If we define $p_n^{-1}(x)$ so that $\mathscr{D}\big(p_{n+1}^{-1}\big)(x) = p_n(x)$. Then

(2RE) $p_n(x) = \widehat{\alpha}_n(\mathscr{D}p_{n+1})(x) + \widehat{\beta}_n(\mathscr{D}p_n)(x) \quad \Rightarrow \quad p_n^{-1}(x) = \widehat{\mathbf{a}}_n p_n(x) + \widehat{\mathbf{b}}_n p_{n-1}(x)$

(1RE) $\Phi(x)p_n(x) = \widetilde{b}_n p_{n+1}^{-1}(x) + \widetilde{c}_n p_n^{-1}(x), \quad \widetilde{c}_n \neq 0$

Nice result by considering all the possible families

Let $\mathbf{u}$ be a Laguerre-type classical linear functional, and let p_n be an ops($\mathbf{u}$). Then:

(TTRR)≡(1RE)

This result implies

$$(x-c)p_n(x) = \mathbf{a}_n\left(\widehat{\mathbf{a}}_{n+1}p_{n+1}(x) + \widehat{\mathbf{b}}_{n+1}p_n(x)\right) + \mathbf{b}_n\left(\widehat{\mathbf{a}}_n p_n(x) + \widehat{\mathbf{b}}_n p_{n-1}(x)\right)$$

Duality

After some renormalization we obtain

(2RE) $p_n^{-1}(x) = p_n(x) - p_{n-1}(x)$

(RR) $(x-c)p_n(x) = \alpha_n\left(p_{n+1}(x) - p_n(x)\right) - \gamma_n\left(p_n(x) - p_{n-1}(x)\right), \quad \gamma_n = \alpha_{n-1}\frac{d_n^2}{d_{n-1}^2}$

The previous expression can be written as

$$\begin{aligned}(x-c)p_n(x) =& d_n^2\left(\frac{\alpha_n}{d_n^2}\Delta_n p_n(x) - \frac{\alpha_{n-1}}{d_{n-1}^2}\nabla_n P_n(x)\right)\\ =& d_n^2\,\nabla_n\frac{\alpha_n}{d_n^2}\Delta_n p_n(x)\\ =& d_n^2\,\Delta_n\frac{\gamma_n}{d_n^2}\nabla_n p_n(x),\end{aligned}$$

where $\Delta_n\big[f(n;x)\big] = f(n+1;x) - f(n;x)$, and $\nabla_n\big[f(n;x)\big] = \Delta_n\big[f(n-1;x)\big]$.

Sturm-Liouville type difference equation

Let $\mathbf{u}$ be a Laguerre-type classical linear functional, and let p_n be an ops($\mathbf{u}$). Then:

$$\gamma_n\,\Delta_n\nabla_n\,p_n(x) + (\alpha_n - \gamma_n)\,\Delta_n\,p_n(x) = (x-c)p_n$$

or

$$\alpha_n\,\Delta_n\nabla_n\,p_n(x) + (\alpha_n - \gamma_n)\,\nabla_n\,p_n(x) = (x-c)p_n$$

No Laguerre-type cases

This process can be done as well but it takes longer to realize it.

Some examples and its Duality

- From the Laguerre polynomials:

$$(n+\alpha)\,\Delta_n\nabla_n y_x(n) + (1-\alpha)\,\Delta_n y_x(n) + x y_x(n) = 0.$$

By using duality

$$(x+\alpha)\,\Delta\nabla y_n(x) + (1-\alpha)\,\Delta y_n(x) + n y_n(x) = 0.$$

If there exists a solution, it can not be a classical one.

- From the Charlier polynomials:

$$n\,\Delta_n\nabla_n y_x(n) + (a-n)\,\Delta_n y_x(n) + x y_x(n) = 0.$$

By using duality

$$x\,\Delta\nabla y_n(x) + (a-x)\,\Delta y_n(x) + n y_n(x) = 0.$$

Solution: The Charlier polynomials are self-dual polynomials.

- From the little q-Jacobi polynomials: let $q \in \mathbb{C}$, $0 < |q| < 1$

$$C_n\Delta_n\nabla_n y_x(n) + (A_n - C_n)\,\Delta_n y_x(n) + q^x y_x(n) = 0$$

where

$$A_n = q^n\frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = aq^n\frac{(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

By using duality

$$C_x\Delta\nabla y_n(x) + (A_x - C_x)\,\Delta y_n(x) + q^n y_n(x) = 0$$

A solution for such a case is unknown yet. And for many other of suck a kind.

Some results

New Discrete Rodrigues-type formula for such families

For the Laguerre polynomials:

$$(-x)^n L_n^{\alpha}(x) = d_n^2(\alpha)\Delta_n^k\left[\frac{1}{d_n^2(\alpha-k)}L_n^{\alpha-k}(x)\right], \quad k = 2, 3, 4, \ldots$$

where $d_n^2(\lambda)$ is the squared norm of $L_n^{\lambda}(x)$.

Observe if $\alpha - k = -n$ then $L_n^{-n}(x) = x^n$. In these case a nice expression can be obtained, which is somehow equivalent of a Rodrigues-type formula.

These Rodrigues-type formulae can be obtained for all the Laguerre-type: Laguerre, Meixner, Charlier, q-Meixner, q-Charlier, Stieltjes-Wigert, q-Laguerre and little q-Laguerre.

Some new formulae

Most of the algebraic formulae that can be obtained for the classical orthogonal polynomials. From them new expressions for the original ones can be obtained in a fancy way.

Example: Since the dual of big q-Jacobi polynomials are Continuous dual q-Hahn:

$$\begin{aligned}\Delta_n p_n(x;a,b,c;q) =& q_{x-1}\left(\frac{\alpha+\beta+1}{2} + n; \sqrt{ab}q, \sqrt{ab^{-1}}q, \frac{cq}{\sqrt{ab}}; q\right)\\ =& \frac{(1-abq^{2n+2})(x-1)}{(1-aq)(1-cq)q^n}p_n(qx;aq,b,cq;q)\end{aligned}$$

Like this one can be obtained, however its calculation is not easy most of the times.

References

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