Semiclassical quasi-orthogonal polynomials
A general calculus approach

Roberto S. Costas Santos
Dpto. Física y Matemáticas, Universidad of Alcalá, Madrid
roberto.costas@uah.es
Joint work with Francisco Marcellan (UC3M, ICMAT)

OPSFA

Semiclassical orthogonal polynomials

Given a linear functional $\mathbf{u} : \mathbb{P}[x] \to \mathbb{C}$ fulfilling the Pearson type equation

$$\textbf{(PE)} \quad \mathscr{D}\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}, \quad \deg \psi \geq 1,$$

where $\mathscr{D} := \frac{d}{dx}$ in the cont. case, $\mathscr{D} := \Delta$ in the disc. case, $\mathscr{D} := \mathscr{D}_q$ in the q-disc. case.

(Class) $\sigma_{\mathbf{u}}$: Given the set $\mathbb{P}^2_{\mathbf{u}} := \{(\phi, \psi) \in \mathbb{P}^2 \ : \ \mathscr{D}\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}\}$, the class of $\mathbf{u}$ is

$$\sigma_{\mathbf{u}} = \min_{(\phi,\psi)\in\mathbb{P}^2_{\mathbf{u}}} \max\{\deg \phi - 2, \deg \psi - 1\}.$$

If p_n is weakly quasi-orthogonal of order r, $p_n(x) =$qops$(\mathbf{u}; r)$, i,e.,

$$\textbf{(WQO-}r\textbf{)} \quad \begin{cases} \langle \mathbf{u}, p_n p_m \rangle = 0, \text{ if } |n-m| > r \\ \exists\, \Lambda \subset \mathbb{N}_0 \text{ finite non-empty } : \text{ if } n \in \Lambda \ \Rightarrow \ \langle \mathbf{u}, p_n p_{n+r} \rangle = 0. \end{cases}$$

Quasi-orthogonality of $(\mathscr{D}p_{n+1})$ [3,4]

If p_n=qops$(\mathbf{u}; r)$ where $\mathbf{u}$ is semiclassical of class $\sigma_{\mathbf{u}}$, then $\mathscr{D}p_{n+1}$=qops$(\phi\mathbf{u}; r + \sigma_{\mathbf{u}})$.

Given $\phi(x) \in \mathbb{P}[x]$. If $\phi(x) = x - d$, then

$$\mathscr{K}_n(x; \phi(x)) = \Omega_{n,d} \frac{P_{n+1}(x)P_n(d) - P_{n+1}(d)P_n(x)}{x-d},$$

and if $\phi(x) = (x-c)\widetilde{\phi}(x)$, then

$$\textbf{(NK)} \quad \mathscr{K}_n(x; \phi(x)) = \Omega_{n,\phi(x)} \frac{K_{n+1}(x; \widetilde{\phi}(x))K_n(c; \widetilde{\phi}(x)) - K_{n+1}(c; \widetilde{\phi}(x))K_n(x; \widetilde{\phi}(x))}{x-c}.$$

We denote these polynomial families Nested kernels associated to $\phi(x)$ [1] .

If p_n is orthogonal with respect to $\mathbf{u}$ (semiclassical), then

$$\textbf{(1RE)} \quad \phi(x)(\mathscr{D}p_{n+1})(x) = \sum_{\nu=n}^{n+\deg\phi} \widetilde{a}_{n,\nu} P_\nu(x) \qquad a_{n+\deg\phi,\nu} \neq 0$$

$$\textbf{(2RE)} \quad p_n(x) = \sum_{\nu=n-\sigma_{\mathbf{u}}}^{n+1} \widehat{a}_{n,\nu}(\mathscr{D}p_\nu)(x) \qquad \widehat{a}_{n,n+1} \neq 0$$

$$\textbf{(SODE)} \quad \phi(x)\mathscr{D}\mathscr{D}^* p_n(x) + \psi(x)\mathscr{D}p_n(x) = \sum_{\nu=n-\sigma_{\mathbf{u}}}^{n+\sigma_{\mathbf{u}}} \lambda_{n,\nu} p_\nu(x)$$

$$\textbf{SODE)} \quad \phi(x)\mathscr{D}\mathscr{K}_n(x; \phi(x)) + \psi(x)\mathscr{K}_n(x; \phi(x)) = \mu_n \mathscr{K}^{\{-1\}}_{n+1}(x; \phi(x)).$$

Characterization of semiclassical quasi-orthogonal polynomials [1]

which is a natural extension of [2].

What happens in the classical case?

The Laguerre case This family is really important since its recurrence relation characterizes the family, since it is equal to its second structure relation. In fact $\phi(x) = x$, and

$$-\mathscr{D}(L^\alpha_{n+1}(x)) = L^{\alpha+1}_n(x) = \frac{(n+1)!}{(\alpha+1)_n} \frac{L^\alpha_{n+1}(x)L^\alpha_n(0) - L^\alpha_{n+1}(0)L^\alpha_n(x)}{x} = \mathscr{K}^L_n(x; x).$$

The Jacobi case In this case $\phi(x) = x^2 - 1$, so

$$\mathscr{K}^{\alpha,\beta}_n(x; \phi(x)) = \Omega_{n,\alpha,\beta} \frac{\mathscr{K}^{\alpha,\beta}_{n+1}(x; x-1)\mathscr{K}^{\alpha,\beta}_n(-1; x-1) - \mathscr{K}^{\alpha,\beta}_{n+1}(-1; x-1)\mathscr{K}^{\alpha,\beta}_n(x; x-1)}{x+1},$$

where

$$\mathscr{K}^{\alpha,\beta}_n(x; x-1) = \Omega_{n,x-1} \frac{P^{\alpha,\beta}_{n+1}(x)P^{\alpha,\beta}_n(1) - P^{\alpha,\beta}_{n+1}(1)P^{\alpha,\beta}_n(x)}{x-1} = \mathscr{D}\left(P^{\alpha,\beta-1}_{n+1}(x)\right).$$

The derivative of the Jacobi polynomials also can be obtained by using an analogous expression starting with $x-1$ and ending with $x+1$.

Observe that

$$\mathscr{D}\left(P^{\alpha,\beta}_{n+1}(x)\right) = \mathscr{K}^{\alpha,\beta}_n(x; \phi(x)).$$

And for the q-Classical case?

The little q-Laguerre case

$$\mathscr{D}_{1/q}(\phi^*(x)w(x)) = \psi(x)w(x), \quad \& \quad \mathscr{D}_q(\phi(x)w(x)) = \psi(x)w(x).$$

where $\phi(x) = x$, $\phi^*(x) = x(x-1)$, $\psi(x) = x - 1 - aq$, the operator $\mathscr{D}_\lambda$ is defined as

$$\mathscr{D}_\lambda f(x) = \frac{f(\lambda x) - f(x)}{(\lambda - 1)x}, \qquad \lambda \in \mathbb{C}, \ \lambda \neq 1, x \neq 0.$$

$$\mathscr{D}_{1/q}\left(P_{n+1}(x; a|q)\right) = \Omega_{n,a,q} \frac{\mathscr{K}^a_{n+1}(x; x)\mathscr{K}^a_n(1; x) - \mathscr{K}^a_{n+1}(1; x)\mathscr{K}^a_n(x; x)}{x-1} = \mathscr{K}^a_n(x; \phi^*(x)),$$

where

$$\mathscr{K}^a_n(x; x) = \Omega_{n,x} \frac{P_{n+1}(x; a|q)P_n(0; a|q) - P_{n+1}(0; a|q)P_n(x; a|q)}{x} = \mathscr{D}_q\left(P_{n+1}(x; a|q)\right).$$

Observe that the last identity holds true because $\phi(x) = x$.

These identities can be obtained for all the families belonging to the Hahn tableaux.

The N-Askey-Wilson polynomials

Let $N \geq 2$, we consider the set with $2N$ parameters

$$S_N := \left\{a^{(N)}_1, a^{(N)}_2, \ldots, a^{(N)}_{2N}\right\}, \quad a^{(N)}_i \in \mathbb{C}, \quad i = 1, 2, \ldots, 2N.$$

We define the N-Askey-Wilson polynomials, $(p_n(x(z); S_N|q))$, as the polynomial eigenfunctions of the second order hypergeometric-type homogeneous linear difference operator

$$\mathscr{H}^{(N)}[y] \stackrel{\text{def}}{=} \frac{1}{\nabla x_1(z)}\left[\phi^{(N)}(-z)\frac{\Delta[y]}{\Delta x(z)} - \phi^{(N)}(z)\frac{\nabla[y]}{\nabla x(z)}\right],$$

on the lattice $x(z) = (q^z + q^{-z})/2$, where $x_1(z) = x(z + 1/2)$,

$$\phi^{(N)}(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})q^{-Nz}\prod_{j=1}^{2N}(q^z - a^{(N)}_j),$$

and $\Delta f(z) = f(z+1) - f(z)$, and $\nabla f(z) = x(z) - x(z-1)$.

The Pearson-type equation

The function $\rho^{(N)}(z)$ defined as

$$\rho^{(N)}(z) \stackrel{\text{def}}{=} -i^N \frac{\big(h(z,1)h(z,-1)\big)^{\frac{N}{2}}\big(h(z,q^{\frac{1}{2}})h(z,-q^{\frac{1}{2}})\big)^{\frac{N}{2}}}{\pi(q^z - q^{-z})^{\frac{N}{2}}\prod_{j=1}^{2N} h(z, a^{(N)}_j)},$$

is a solution of the Pearson-type difference equation

$$\Delta\left(\mathbb{X}(z)\phi^{(N)}(z)\right) = \psi^{(N)}(z)\mathbb{X}(z)\nabla x_1(z),$$

where $h(z,a) = \prod_{k=0}^{\infty}\left(1 - 2ax(z)q^k + a^2q^{2k}\right)$,

$$\psi^{(N)}(z) = \frac{\phi^{(N)}(-z) - \phi^{(N)}(z)}{\nabla x_1(z)} = 2^N(a^{(N)}_1 a^{(N)}_2 \cdots a^{(N)}_{2N} - 1)(x(z))^{N-1} + \cdots,$$

and

$$\widehat{\phi}^{(N)}(z) \stackrel{\text{def}}{=} \phi^{(N)}(z) + \frac{1}{2}\psi^{(N)}(z)\nabla x_1(z) = 2^{N-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(a^{(N)}_1 a^{(N)}_2 \cdots a^{(N)}_{2N} + 1)(x(z))^N + \cdots.$$

The hypergeometric representation [1]

The N-Askey-Wilson polynomials can be written as follows

$$P_n(x(z); S_N|q) = B_{n,N}\cdot {}_{2N+4}\phi_{2N+3}\left(\begin{matrix} a, q^{-n}, a^{\frac{1}{2}}q, -a^{\frac{1}{2}}q, \alpha_1, \ldots, \alpha_{2N} \\ aq^{n+1}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/\alpha_1, \ldots, aq/\alpha_{2N} \end{matrix} \,\middle|\, q; \frac{(aq)^N q^n}{\prod_{i=1}^{2N}\alpha_i}\right),$$

where $B_{n,N}$ is a non-zero constant, $a = q^{2z-N}$, and $\alpha_i = q^z a_i$, $i = 1, 2, \ldots, 2N$.

We obtained orthogonality, another representations, Rodrigues formula, etc..

References

[1] Costas-Santos, R. S. and Marcellán F. *Characterization of semiclassical quasi-orthogonal polynomials*, 2019. In progress.

[2] Costas-Santos, R. S. and Marcellán F. q-Classical Orthogonal Polynomials: A General Difference Calculus Approach. *Acta Appl Math* **111** (2010), 107–128.

[3] Hendriksen, E. and Van Rossum, H. Semi-classical orthogonal polynomials. In '*Polynômes Orthogonaux et Applications*' (C. Brezinski et al., eds.), Springer-Verlag, Berlin. *Lecture Notes in Math.* **1171** (1985), 354–361.

[4] Shohat, J.A. A differential equation for orthogonal polynomials. *Duke Math. J.* **5** *(1939), 401–417.*