Recent Trends in Constructive Approximation Theory

Classical orthogonal polynomials
A general difference calculus approach

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Menu

1. Overview of Classical orthogonal polynomials
2. Connection with Operator Theory. The Rodrigues operator
3. More general Hahn Theorem
4. New Theorem of Characterization
5. An example: The Askey-Wilson Polynomials

Ingredients: The Classical Orthogonal Polynomials

Standard and Δ

1. Standard classical orthogonal polynomials (Hermite, Laguerre, Jacobi)

$$> \quad \mathfrak{H} := \tilde{\sigma}(x)\frac{d^2}{dx^2} + \tilde{\tau}(x)\frac{d}{dx}, \qquad \lambda_n = n\tilde{\tau}' + n(n-1)\frac{\tilde{\sigma}''}{2}.$$

$$> \quad \frac{d}{dx}[\tilde{\sigma}(x)\rho(x)] = \tilde{\tau}(x)\rho(x).$$

2. Δ-classical orthogonal polynomials (Hahn, Meixner, Kravchuk, Charlier, etc)

$$> \quad \mathfrak{H}_\Delta := \sigma(s)\Delta\nabla + \tau(s)\Delta, \qquad \lambda_n = n\tilde{\tau}' + n(n-1)\frac{\tilde{\sigma}''}{2}.$$

$$> \quad \sigma(x) := \tilde{\sigma}(x) - \frac{1}{2}\tilde{\tau}(x), \tau(s) = \tilde{\tau}(x),$$

$$> \quad \Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)$$

$$> \quad \Delta f(s) = f(s+1) - f(s), \nabla f(s) = f(s) - f(s-1),$$

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q-Polynomials

3. q-classical orthogonal polynomials (or q-Polynomials)

> $\mathfrak{H}_q = \sigma(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)} + \tau(s)\frac{\Delta}{\Delta x(s)}$, $x_k(s) = x(s+\frac{k}{2})$,

> $\sigma(s) := \tilde{\sigma}(x(s)) - \frac{1}{2}\tilde{\tau}(x(s))\nabla x_1(s)$, $\tau(s) = \tilde{\tau}(x(s))$,

> $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\nabla x_1(s)$,

> $x(s) = c_1 q^s + c_2 q^{-s} + c_3$.

Polynomial eigenfunctions of $\mathfrak{H}_q$

$$P_n(s)_q := \left[\frac{B_n \nabla \rho_1(s)}{\rho_0(s)\nabla x_1(s)}\right]\left[\frac{\nabla \rho_2(s)}{\rho_1(s)\nabla x_2(s)}\right]\cdots\left[\frac{\nabla \rho_n(s)}{\rho_{n-1}(s)\nabla x_{n-1}(s)}\right],$$

Symmetric form of $\mathfrak{H}_q$

$$\mathfrak{H}_q = \left[\frac{1}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\rho_1(s)\right]\frac{\Delta}{\Delta x(s)}.$$

A foretaste: Connection with Operator Theory

The Rodrigues Operator

Definition 1 Given functions σ and ρ, where ρ is supported on Ω, and a lattice $x(s)$, we define the k**-th Rodrigues operator associated with** $(\sigma(s), \rho(s), x(s))$ as

$$R_0(\sigma,\rho,x) := I,\ R_1(\sigma,\rho,x) := \frac{\nabla\rho_1(s)}{\rho(s)\nabla x_1(s)}\circ I,$$

$$R_k(\sigma,\rho,x) := R_1(\sigma(s),\rho(s),x(s))\circ R_{k-1}(\sigma(s),\rho_1(s),x_1(s)),$$

where $\rho_1(s) = \sigma(s+1)\rho(s+1)$ and I is the identity operator.

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Standard COP: $R_1(\sigma,\rho) := \dfrac{1}{\rho(x)}\dfrac{d\rho_1(s)}{dx} \circ I,\ \rho_1(x) := \rho(x)\tilde{\sigma}(x).$

Δ-COP: $R_1(\sigma,\rho) := \dfrac{\nabla\rho_1(s)}{\rho(s)} \circ I,\ \rho_1(s) := \rho(s+1)\sigma(s+1).$

Some well known results

- Menu

Ingredients: The Classical Orthogonal Polynomials

A foretaste: Connection with Operator Theory

- The Rodrigues Operator
- Some well known results

The Chef's Special: The Main Results

The dessert

1. $R_1(\sigma, \rho, x)[1] = \tau(s)$. *q*-Pearson equation.
2. The operators $\Delta^{(1)}$ and $R_1(\sigma, \rho, x)$ are lowering and raising operators associated to the q-hamiltonian $\mathfrak{H}_q$, respectively. In fact,
$$\mathfrak{H}_q = R_1(\sigma, \rho, x)\Delta^{(1)} \circ I.$$
3. For every integers, $n,\ k$, $0 \le k \le n$, there exists a constant, $C_{n,k}$ such that
$$\Delta^{(k)} P_n(s)_q = C_{n,k} R_{n-k}(\sigma, \rho_k, x_k)[1].$$

Where $x_k(s) := x(s + \frac{k}{2})$, $\rho_k(s) := \rho_{k-1}(s+1)\sigma(s+1)$, being $\rho_0 \equiv \rho$, and
$$\Delta^{(k)} := \begin{cases} \dfrac{\Delta}{\Delta x_{k-1}} \dfrac{\Delta}{\Delta x_{k-2}} \cdots \dfrac{\Delta}{\Delta x}, & \text{if } k \ge 0, \\ R_k(\sigma, \rho_k, x_k), & \text{if } k \le 0. \end{cases}$$

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The Chef's Special: The Main Results

New Hahn's Theorem

Theorem 1 *Let $\{P_n\}_{n\geq 0}$ be an OPS with respect to $\rho(s)$ such that is complete as orthonormal set in $\ell^2([a,b],\langle\cdot,\cdot\rangle_\rho)$. The following statements are equivalent.*

(i) $\{P_n\}_{n\geq 0}$ *is q-classical and the following boundary conditions hold*

$$x^k(s)x_{-1}(s)^l\sigma(s)\rho(s)\Big|_{s=a}^{s=b}=0,\quad k,l=0,1,\ldots(*)$$

(ii) $\{\Delta^{(1)}P_{n+1}\}_{n\geq 0}$ *is an OPS with respect to $\widetilde{\rho}(s)$ and the following boundary conditions hold*

$$x^k(s)x_{-1}(s)^l\widetilde{\rho}(s-1)\Big|_{s=a}^{s=b}=0,\quad k,l=0,1,\ldots$$

q-classical OPS

Definition 2 The sequence $\{P_n\}_{n\geq 0}$ is said to be a **q-classical OPS** on the lattice $x(s)$ if satisfies the orthogonality conditions

$$\sum_{s=a}^{b-1} P_n(s)P_m(s_i)\rho(s)\nabla x_1(s) = d_n^2\delta_{n,m},\ \ \Delta s = 1,\ \ n,m = 0,1,\ldots$$

where

(i) $\rho(s)$ is a solution of the q-Pearson equation $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\nabla x_1(s)$.

(ii) $\sigma(s) + \frac{1}{2}\tau(s)\nabla x_1(s)$ is a polynomial on $x(s)$ of degree, at most, 2.

(iii) $\tau(s)$ is a polynomial on $x(s)$ of degree 1.

q-numbers

$$[s]_q := \frac{q^{\frac{s}{2}} - q^{-\frac{s}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad q \in \mathbb{C}, |q| \neq 1.$$

New Theorem of Characterization

Theorem 2 *Let $\{P_n\}_{n\geq 0}$ be an OPS with respect to $\rho(s)$ on the lattice $x(s)$ and let $\sigma(s)$ be such that boundary condition (*) holds. Then the following statements are equivalent:*

1. $\{P_n\}_{n\geq 0}$ *is a q-classical OPS.*

2. *The sequence $\{\Delta^{(1)} P_n\}_{n\geq 0}$ is an OPS with respect to $\rho_1(s)$ where ρ satisfies the last q-Pearson equation.*

3. *For every integer k, the sequence $\{R_n(\rho_k(s), x_k(s))[1]\}_{n\geq 0}$ is an OPS with respect to the weight function $\rho_k(s)$ where $\rho_0(s) = \rho(s)$, $\rho_k(s) = \rho_{k-1}(s+1)\sigma(s+1)$ and ρ satisfies last q-Pearson equation.*

4. *(Second order difference equation):*

$$\sigma(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)} + \lambda_n P_n(s) = 0,$$

New Theorem of Characterization (cont.)

5. $\{P_n\}_{n\geq 0}$ can be expressed in terms of the Rodrigues operator

$$P_n(s) = B_n R_n(\rho(s), x(s))[1] = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x_1(s)} \cdots \frac{\nabla[\rho_n(s)]}{\nabla x_n(s)},$$

6. (First structure relation):

$$\phi(x_1(s)) \frac{\Delta P_n(s)}{\Delta x(s)} = a_n \mathcal{M} P_{n+1}(s) + b_n \mathcal{M} P_n(s) + c_n \mathcal{M} P_{n-1}(s) + j_n x_1(s) \mathcal{M} P_n(s).$$

7. (Second structure relation):

$$\mathcal{M} P_n(s) := \frac{P_n(s+1) + P_n(s)}{2} = e_n \frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n \frac{\Delta P_n(s)}{\Delta x(s)} + g_n \frac{\Delta P_{n-1}(s)}{\Delta x(s)},$$

where $e_n \neq 0$, $g_n \neq \gamma_n$ for all $n \geq 0$, and γ_n is the corresponding coefficient of the following three-term recurrence relation $x(s) P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s)$.

The dessert

The Askey-Wilson polynomials

These polynomials are the polynomial eigenfunctions of the second order linear difference operator:

$$\mathfrak{H}_q = \frac{1}{\nabla x_1(s)} \left(\sigma^{AW}(-s) \frac{\Delta}{\Delta x(s)} - \sigma^{AW}(s) \frac{\nabla}{\nabla x(s)} \right),$$

where $x(s) = \frac{1}{2}(q^s + q^{-s})$,

$$\sigma^{AW}(s) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 q^{-2s+\frac{1}{2}}(q^s - a)(q^s - b)(q^s - c)(q^s - d),$$

with eigenvalues $\lambda_n = 4q^{1-n}(1-q^n)(1-abcdq^{n-1})$.
Let the function

$$\rho(s) := q^{-2s^2}(a,b,c,d;q)_s(a,b,c,d;q)_{-s}.$$

where $(a;q)_0 := 1$, $(a;q)_k = (1-a)(aq;q)_{k-1}$, $k > 0$,
$(1-aq^{-1})(a;q)_k = (aq^{-1};q)_{k+1}$, $k < 0$.

The Askey-Wilson polynomials (cont.)

1. $\phi^{AW}(s) = \frac{1}{2}\left(\sigma^{AW}(s+1) + \sigma^{AW}(-s)\right) = q_2(x_1(s))$,
2. $a_n = -2q^{\frac{1}{2}-n}\frac{1-abcdq^{n-1}}{1-abcdq^{2n}}\left(\frac{2(1-q^{n+1})(1-abcdq^{n-1})}{1-abcdq^{2n-1}} - 1 + q\right)$,
3. $c_n = 4q^{\frac{1}{2}}\frac{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}{1-abcdq^{n-2}}(1-abcdq^{n-1})(1-q^{n+1})A_{n-1}C_n$,
4. $e_n = \frac{2[n]_q(1-abcdq^{n-1})^2 - [2]_q[n+1]_q(1-abcdq^n)^2}{4[n]_q(1-abcdq^{n-1})(1-abcdq^{2n-1})(1-abcdq^{2n})}$,
5. $f_n = \frac{1-q}{4}(a - a^{-1}q^{-1}) - \frac{1}{2}\left(A_n + C_n - \frac{[2]_q}{2}(\widetilde{A}_{n-1} + \widetilde{C}_{n-1})\right)$,

where

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

The coffee: Relevant references

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Finally ....

THANKS FOR YOUR ATTENTION !!