Extensions of discrete classical orthogonal polynomials beyond the orthogonality

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Structure of the talk

Basic properties

- Let (P_n) be a polynomial sequence and u be a linear functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathcal{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1,$$

where

$$\mathcal{D} = \frac{d}{dx}, \text{ or } \mathcal{D} = \nabla, \text{ or } \mathcal{D} = \frac{\nabla}{\nabla x(s+1/2)}.$$

- Three-term recurrence relation:

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

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The families

- Continuous Classical OP: Jacobi, Hermite, Laguerre and Bessel.
- Δ-Classical OP: Hahn, Racah, Meixner, Krawtchouk, Charlier, etc.
- q-Classical OP: Askey Wilson, q-Racah, q-Hahn, Continuous q-Hahn, Big q-Jacobi, q-Hermite, q-Laguerre, Al-Salam-Chihara, Stieltjes-Wigert, etc.

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$$h_n^{\alpha,\beta}(x;N) = \frac{(-N,\alpha+1)_n}{(\alpha+\beta+n+1)_n} {}_3F_2\left(\begin{matrix} -n, \alpha+\beta+n+1, -x \\ -N, \alpha+1 \end{matrix} \middle| 1 \right).$$

- Property of orthogonality.

$$\langle \mathbf{u}^H, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\Delta(x(\beta+N+1-x)\mathbf{u}^H) = ((\alpha+1)N - (\alpha+\beta+2)x)\mathbf{u}^H.$$

- Integral representation with some boundary condition:

$$\langle \mathbf{u}^H, P \rangle = \sum_{x=0}^{N} P(x) \frac{\Gamma(\beta+N+1-x)\Gamma(\alpha+1+x)}{\Gamma(1+x)\Gamma(N+1-x)}.$$

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Continuous Hahn polynomials

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$$p_n(x;a,b,c,d) = D_n\, {}_3F_2\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{array} \middle| 1 \right).$$

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The limit relation H-cH

- The hypergeometric serie, re-written:

$$h_n^{\alpha,\beta}(x;N) = r_n \sum_{k=0}^{n} \frac{(-n, \alpha+\beta+n+1, -x)_k (-N+k)_{n-k}}{(\alpha+1, 1)_k},$$

- The limit relation:

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About the zeros of the extended Hahn polynomials

Figure: Zeros of $h_{15}^{1,1}(x;5)$ (left) and $h_{15}^{1,15}(x;5)$ (right)

The factorization

- For any integer k, $0 \leq k \leq n$,

$$\Delta^k h_n^{\alpha,\beta}(x;N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x;N-k),$$

$$ii.2) \qquad \nabla^k h_n^{\alpha,\beta}(x;N) = (n-k+1)_k h_{n-k}^{\alpha+k,\beta+k}(x-k;N-k).$$

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$$h_n^{\alpha,\beta}(x;N) \quad = (x-N)_{N+1}(-i)^{n-N-1}p_{n-N-1}(ix;N+1,\beta+N+1,1,$$
$$\alpha+1) = (x-N)_{N+1}(-i)^{n-N-1}p_{n-N-1}\left(\left(x-\tfrac{N}{2}\right)i;1+\tfrac{N}{2},\beta+1+\right.$$
$$\left.+\tfrac{N}{2},1+\tfrac{N}{2},\alpha+1+\tfrac{N}{2}\right).$$

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A characterization Theorem for the Hahn polynomials

Theorem: Let N be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that: $-\alpha, -\beta \notin \{1, 2, \dots, N, N+2, \dots\}$, and $-\alpha-\beta \notin \{1, 2, \dots, 2N+1, 2N+3, \dots\}$. Then the family of Hahn polynomials is a OPS with respect to the Δ-Sobolev inner product:

$$(f,g)_S = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha,\beta}(x;N) + \int_C (\Delta^{N+1} f(z))(\Delta^{N+1} g(z))\omega^{\alpha,\beta}(z;N)dz,$$

where

$$\begin{aligned} \rho^{\alpha,\beta}(x;N) &= \frac{\Gamma(\beta+N+1-x)\Gamma(\alpha+x+1)}{\Gamma(N+1-x)\Gamma(x+1)}, \\ \omega^{\alpha,\beta}(z;N) &= \Gamma(-z)\Gamma(\beta+N+1-z)\Gamma(1+z)\Gamma(\alpha+N+2+z), \end{aligned}$$

and C is a complex contour from $-\infty i$ to ∞i which separates the poles of the functions $\Gamma(-z)\Gamma(\beta+N+1-z)$ and $\Gamma(1+z)\Gamma(\alpha+N+2+z)$.

Wilson $\longrightarrow$ Racah

- Factorization: If $\alpha + 1 = -N$ we get

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = R_{N+1}(\lambda(x);-N-1,\beta,\gamma,\delta)(-1)^{n-N-1}$$
$$\times W_{n-N-1}\left(\left(i\left(x+\frac{\gamma+\delta+1}{2}\right)\right)^2; N+\frac{\gamma+\delta+3}{2}, \frac{-\gamma-\delta+1}{2}, \beta+\frac{-\gamma+\delta+1}{2}, \frac{\gamma-\delta+1}{2}\right).$$

- The Δ-Sobolev orthogonality:

$$\langle p,q\rangle_S = \langle p,q\rangle_d + \left\langle (\Delta/\Delta\lambda)^{N+1} p, (\Delta/\Delta\lambda)^{N+1} q \right\rangle_c,$$

with

$$\langle p,q\rangle_d = \sum_{x=0}^{N} p(x)q(x)\frac{(\alpha+1,\beta+\delta+1,\gamma+1,\gamma+\delta+1,(\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1,-\beta+\gamma+1,(\gamma+\delta+1)/2,\delta+1,1)_x},$$
$$\langle p,q\rangle_c = \int_C p(z^2)q(z^2)\nu(zi+i+i(\gamma+\delta+N)/2)\nu(-(zi+i+i(\gamma+\delta+N)/2))dz.$$

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- Continuous Dual Hahn polynomials $\longrightarrow$ Dual Hahn polynomials.
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The limits relations between the families

- Racah $\rightarrow$ Hahn.

$$\lim_{\delta\to\infty} R_n(\lambda(x); -N-1, \beta+\gamma+N+1, \gamma, \delta) = h_n^{\gamma,\beta}(x;N).$$

- Racah $\rightarrow$ Dual Hahn.

$$\lim_{\beta\to\infty} R_n(\lambda(x); -N-1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).$$

- Hahn $\rightarrow$ Krawtchouk.

$$\lim_{t\to\infty} h_n^{(1-p)t,pt}(x;N) = K_n(x;p,N).$$

- Dual Hahn $\rightarrow$ Krawtchouk.

$$\lim_{t\to\infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x;p,N).$$

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Orthogonality relations for Meixner polynomials with general parameter

The Meixner polynomials and continuous Hahan polynomials are related through the following limit relation:

$$\lim_{|t|\to\infty}(-i)^n p_n(ix;0,-t/c,t,\beta) = M_n(x;\beta,c), \qquad n=0,1,2,\dots.$$

Proposition: For any $\beta, c \in \mathbb{C}$, $c \notin [0,\infty)$ and $-\beta \notin \mathbb{N}$, the following property of orthogonality for the Meixner polynomials fulfills:

$$\int_C M_n(z;c,\beta) z^m \Gamma(-z)\Gamma(\beta+z)(-c)^z dz = 0,\ 0 \leq m < n,\ n=0,1,2,\dots,$$

where C is a complex contour from $-\infty i$ to ∞i separating the increasing poles $\{0,1,2,\dots\}$ from the decreasing poles $\{-\beta,-\beta-1,-\beta-2,\dots\}$.

The examples considered (under construction)

q-Hahn,
q-Racah,
dual q-Hahn,
quantum q-Krawtchouk,
q-Krawtchouk,
affine q-Krawtchouk, and
dual q-Krawtchouk polynomials.

Basic properties

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$$h_n^{\alpha,\beta}(x; N; q) = \frac{(q^{-N}, q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \, {}_3\phi_2\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, x \\ q^{-N}, q^{\alpha+1} \end{array} \middle| \, q; q \right).$$

- Property of orthogonality.

$$\langle \mathbf{u}^{qH}, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathcal{D}_q(q^{\alpha}(q^{\beta+1+x} - q^{-N})\mathbf{u}^{qH}) = h_1^{\alpha,\beta}(x; N; q)\mathbf{u}^{qH}.$$

- Integral representation with some boundary condition:

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- Property of orthogonality.

$$\langle \mathbf{u}^{qH}, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathcal{D}_q(q^{\alpha}(q^{\beta+1+x} - q^{-N})\mathbf{u}^{qH}) = h_1^{\alpha,\beta}(x; N; q)\mathbf{u}^{qH}.$$

- Integral representation with some boundary condition:

$$\langle \mathbf{u}^{qH}, P \rangle = \sum_{x=0}^{N} P(x) \frac{(q^{\alpha+1}, q^{-N}; q)_x}{(q, q^{-\beta-N}; q)_x} \, q^{-(\alpha+\beta)x}.$$

The big q-Jacobi

- Basic Hypergeometric series:

$$P_n(x;a,b,c;q) = \frac{(q^{a+1},q^{c+1};q)_n}{(q^{a+b+n+1};q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{a+b+n+1}, x \\ q^{a+1}, q^{c+1} \end{matrix} \middle| q;q \right).$$

- Property of orthogonality: $\langle \mathbf{u}^{BqJ}, P_n P_m \rangle = d_n^2 \delta_{n,m}$.
- Distributional equation:

$$\frac{\Delta}{\Delta x}((x-aq)(x-cq)\mathbf{u}^{BqJ}) = P_1(x;a,b,c;q)\mathbf{u}^{BqJ}.$$

- Integral representation with some boundary condition:

$$\langle \mathbf{u}^{BqJ}, P \rangle = \int_{cq}^{aq} P(x) \frac{(q^{-a}x, q^{-c}x;q)_\infty}{(x, q^{b-c}x;q)_\infty} d_q x.$$

The big q-Jacobi

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$$P_n(x;a,b,c;q) = \frac{(q^{a+1}, q^{c+1};q)_n}{(q^{a+b+n+1};q)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n}, q^{a+b+n+1}, x \\ q^{a+1}, q^{c+1} \end{array}\,\middle|\, q;q\right).$$

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The limit relation *BqJ-qH*

- The Basic hypergeometric series, re-written:

$$h_n^{\alpha,\beta}(x;N;q) = \frac{(q^{\alpha+1};q)_n}{(q^{\alpha+\beta+n+1};q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, q^{\alpha+\beta+n+1}, q^{-x};q)_k (q^{-N+k};q)_{n-k}}{(q^{\alpha+1}, q; q)_k} \, q^k .$$

- The key...the following algebraic relation:

$$\frac{(q^{j+1};q)_{n-N-1-j}(q^{-n};q)_{N+1+j}}{(q;q)_{N+1+j}} = \frac{(q;q)_{n-N-1}(q^{-n};q)_{N+1}(q^{-n+N+1};q)_j}{(q;q)_{N+1}(q^{N+2},q;q)_j} .$$

- The limit relation:

$$h_n^{\alpha,\beta}(x;N;q) = \lim_{\varepsilon \longrightarrow 0} P_n(x;\alpha,\beta,-N-1+\varepsilon;q).$$

The limit relation *BqJ-qH*

- The Basic hypergeometric series, re-written:

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- The limit relation:

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About the zeros of the extended q-Hahn polynomials

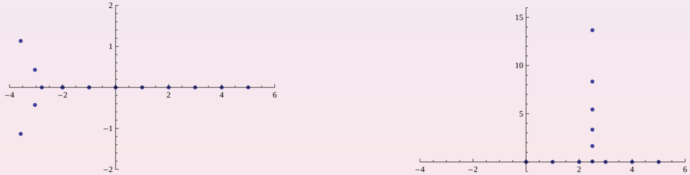

Figure: Log of the zeros of $h_{15}^{1,1}(x;5;0.5)$ (left) and $h_{15}^{1,1}(x;5;\exp(0.23I))$ (right)

The factorization

- For any integer k, $0 \leq k \leq n$,

$$\mathcal{D}^k_{q^{-1}} h_n^{\alpha,\beta}(x;N;q) = (n-k+1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x;N-k;q),$$
$$\mathcal{D}^k_{q} h_n^{\alpha,\beta}(x;N;q) = (n-k+1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x-k,N-k;q),$$

- The factorization:

$$h_n^{\alpha,\beta}(q^{-x};N;q) = \frac{h_{N+1}^{\alpha,\beta}(q^{-x};N;q)}{q^{(N+1)(n-N-1)}} P_{n-N-1}(-x+N+1;\alpha+N+1,\beta+N+1,N+1;q).$$

The factorization

- For any integer k, $0 \leq k \leq n$,

$$\mathcal{D}_{q^{-1}}^{k} h_n^{\alpha,\beta}(x;N;q) = (n-k+1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x;N-k;q),$$
$$\mathcal{D}_{q}^{k} h_n^{\alpha,\beta}(x;N;q) = (n-k+1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x-k,N-k;q),$$

- The factorization:

$$h_n^{\alpha,\beta}(q^{-x};N;q) = \frac{h_{N+1}^{\alpha,\beta}(q^{-x};N;q)}{q^{(N+1)(n-N-1)}} P_{n-N-1}(-x+N+1;\alpha+N+1,\beta+N+1,N+1;q).$$

A characterization Theorem for the q-Hahn polynomials

Theorem: Let N be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that: $-\alpha, -\beta \notin \{1, 2, \ldots, N, N+2, \ldots\}$, and $-\alpha-\beta \notin \{1, 2, \ldots, 2N+1, 2N+3, \ldots\}$. Then the family of Hahn polynomials is a OPS with respect to the $\mathcal{D}_{q^{-1}}$-Sobolev inner product:

$$(f, g)_S = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha,\beta}(x; N; q) + \int_C (\mathcal{D}_{q^{-1}}^{N+1} f(z))(\mathcal{D}_{q^{-1}}^{N+1} g(z))\omega^{\alpha,\beta}(z; N; q)dz$$

where

$$\rho^{\alpha,\beta}(x; N; q) = \frac{\Gamma_q(\alpha+1+x)\Gamma_q(x-N)}{\Gamma_q(1+x)\Gamma_q(-\beta-N+x)} q^{-(\alpha+\beta)x},$$

and C is a complex contour from $-\infty i$ to ∞i which separates the certain poles*.

Some references

- R S Costas-Santos, and J F Sánchez-Lara 2008 (In press) *Extensions of discrete classical orthogonal polynomials beyond the orthogonality*. JCAM 2008.
- R S Costas-Santos, and J F Sánchez-Lara 2008 (In prep.) *q-Extensions of discrete classical orthogonal polynomials beyond the orthogonality*. Available in my web soon.

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- R S Costas-Santos, and J F Sánchez-Lara 2008 (In prep.) *q-Extensions of discrete classical orthogonal polynomials beyond the orthogonality*. Available in my web soon.

Finally...

Thank for your attention!!